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A multi-scaling analysis of the surface of a three-dimensional pile of rice

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Abstract. – The scaling properties of the surface of a three-dimensional rice pile are studied. In time, the pile surface exhibits multi-scaling, where the exponents decrease with increasing moment. This is connected to the fact that the pile is in a self-organized critical state, where the dynamics of the steady state is influenced by transient properties, leading to multi-scaling. In contrast, when studying the transient properties directly, generic scaling is observed, as there is only one relevant cut-off. Similarly in space, multi-scaling is absent and within experimental error, the same scaling is observed for all moments. Again, this is connected to the properties of the critical state.

Introduction. – The difference between self-organized criticality (SOC) [1] and ordinary non-equilibrium critical systems [2] has been a theoretical puzzle for some years. Recently, some advances have been made, which shed light on the connections of SOC to absorbing state phase transitions [3] and extremal dynamics [4]. In both of these cases, for a good understanding it is important to study the transients, which ultimately lead to the stationary, critical state. The separate dynamics of these transients leads, according to Paczuski [5], to an additional correlation length in SOC systems. In the critical state, the usual dynamic correlation length has to be described together with this transient correlation length, in a unified scaling picture, which leads to multi-scaling properties in the temporal behaviour of the critical state. In ordinary critical systems, such as the roughening of a combustion front in paper [6] at longer length and time scales, there is only a single correlation length, corresponding to the critical point, and thus generic scaling. Therefore, it is possible to distinguish between ordinary critical systems that are *tuned* to a critical point (showing generic scaling) and SOC systems, which arrive at the critical point by their internal, extremal dynamics [7] (showing multi-scaling).

Vespigniani *et al.* view [8] SOC as just the special case of a driven absorbing state phase transition, where the driving rate is "tuned" to a critical value of zero. Note, however, that also in the self-organization process they describe, transients appear that obey a different dynamics such that also in this view SOC systems are special in that they have a transient correlation length and hence show multi-scaling behaviour.

Furthermore, in the initial approach to the critical state, the transient state itself, there is only one time scale present, namely that corresponding to the transient dynamics. Therefore the transient state is expected to show generic scaling in time. Thus, a system which shows a critical state out of equilibrium can be tested for SOC by comparing the temporal scaling behaviour of the stationary (critical) and transient states.

Here, we study experimentally the scaling properties of the surface of a pile of rice in both the stationary and the transient states. A pile of rice is particularly suited for this type of study, as SOC behaviour has been observed in both two-dimensional [9] and three-dimensional piles [10]. Moreover, the signatures of SOC that have been observed go beyond the mere power law distribution of avalanches, but include their finite-size scaling, as well as observations of surface roughening [10, 11]. Thus the study of the multi-scaling properties can lead to an experimental confirmation of the fact that SOC systems show a transient correlation length and hence display multi-scaling in time in the stationary state. When studying the build-up to the critical state directly, however, only effects on scales below the transient correlation length are measured and one expects to observe generic scaling in the temporal behaviour of the rice pile.

In addition, we study the spatial scaling behaviour in the *stationary* state of the pile. There, the roughness of the surface is mainly determined by big avalanches, such that there is again only one cut-off scale and thus generic scaling is expected.

Experimental system. – The experimental system consists of a pile of rice with a base area of $1 \times 1\,\mathrm{m}^2$, which has been described in detail elsewhere [10, 12]. The pile is fed continuously from a uniformly distributed line close to the wall of an open box. An experiment consists of ~ 400 images taken every 30 s. The driving is slow, such that between images a volume of ~ 1500 grains is added over the whole line of growth, compared to $\sim 10^8$ grains in the pile. This only corresponds to the addition of 1-2 grains locally. The surface properties are studied using an implementation of monocular stereoscopy, custom-built to serve the purpose of our experiment [13]. In order to obtain accurate reconstructions of the pile surface, a set of 300 coloured lines (red, green and blue) is projected onto the surface and is observed with a high-resolution charge-coupled device camera (2048×1596 pixels) at an angle of 45° to the illumination. The

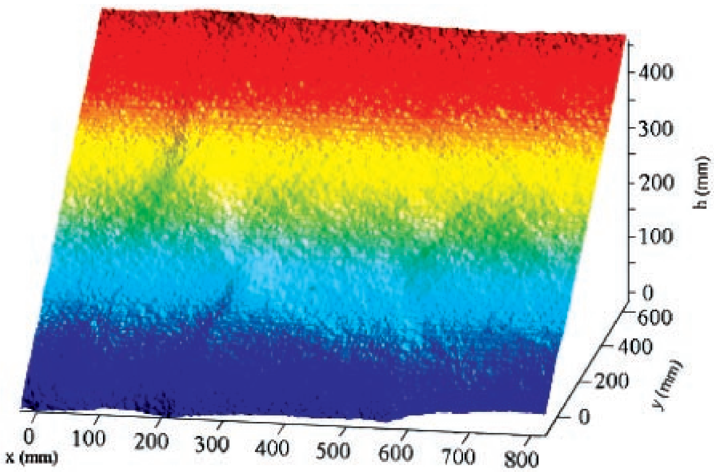

Fig. 1 – Reconstruction of a typical rice pile surface. The surface coordinates are reconstructed using a technique based on the projection of a set of coloured lines to identify lines of equal height. The resolution and accuracy of the reconstruction method are $\sim$ 1-2 mm, comparable to the size of a grain of rice.

surface properties are then obtained from the deformation of the identified lines from simple geometry. With this method, it is possible to reconstruct the whole surface to an accuracy and resolution of 1-2 mm, which is comparable to the size of a grain ($2 \times 2 \times 7\,\mathrm{mm}^3$) [12, 13]. A typical reconstruction of a pile is shown in fig. 1. For the study of the stationary state, the results discussed here come from three separate experiments, while a set of nine experiments was carried out to study the transient behaviour. In the latter case, the pile was prepared in a smooth state at an angle considerably lower than the critical angle ($\phi \simeq 0.55$ rad compared to $\phi_c \simeq 0.8$ rad). The transient period studied here consists of the first 150 images in the evolution of the pile, before the occurrence of avalanches spanning the system size. The experiments were carried out under varying ambient conditions, with consistent results.

Analysis methods. – In the steady state, a system with generic scale invariance, such as a KPZ or similar Langevin roughening system [6, 14], will produce a self-affine interface, which can be described by a single set of roughening and growth exponents, α and β [14]. The roughness exponent describes the roughness of the spatial interface, while the growth exponent describes the evolution of the roughness in time. Classically, the roughness is quantified via the width of the interface (in two dimensions), given by

$$w(t, L) = \left\langle \left(h(\vec{x}, t) - \langle h(t)\rangle_L\right)^2 \right\rangle_L^{1/2}, \tag{1}$$

where $h(\vec{x}, t)$ is the height of the interface, L is the linear system size and $\langle \ldots \rangle_L$ denotes a spatial average over the size of the system. For a self-affine surface, the width scales with time and space as $w(t, L) \propto t^\beta$ for $t < t_\times$ and $w(t, L) \propto L^\alpha$ for $t > t_\times$. Here $t_\times$ is the crossover time, where the correlation length of the roughness reaches the system size [14]. The same scaling is observed using the two-point correlation function,

$$C(\vec{x}, t) = \left\langle \left(h(\vec{\xi}, \tau) - h(\vec{x} + \vec{\xi}, t + \tau)\right)^2 \right\rangle_{\xi,\tau}^{1/2}, \tag{2}$$

where the averaging runs over all $\vec{\xi}$ and τ. Thus, $C(|\vec{x}|, 0) \propto |\vec{x}|^\alpha$ and $C(0, t) \propto t^\beta$. In many cases, $C(\vec{x}, t)$ is more appropriate than the width for a reliable determination of the roughness and growth exponents [14], which is why in the following the correlation function is used.

For a SOC system, however, it is expected that generic scaling does not appear in the time behaviour (see below and [5]). Then the roughness has to be quantified according to the various moments of the surface instead of just the width. In such a study, the bigger fluctuations are weighted more strongly with increasing moment. Thus for $q \gtrsim 20$ only the biggest one or two fluctuations will be counted. In order to study a significant variation of the scaling behaviour with increasing moment, care should be taken that a reasonable number of fluctuations is averaged over in the analysis. Therefore, we determine in the following the q-th–order correlation function in the range of $1 \le q \le 20$. In this range of q, in every experiment, a set of at least 10 fluctuations is averaged over. The q-th–order correlation function is given by

$$C_q(\vec{x}, t) = \left\langle \left|h(\vec{\xi}, \tau) - h(\vec{x} + \vec{\xi}, t + \tau)\right|^q \right\rangle_{\xi,\tau}^{1/q}. \tag{3}$$

Here, $C_q(|\vec{x}|, 0) \propto |\vec{x}|^{\alpha(q)}$ and $C_q(0, t) \propto t^{\beta(q)}$. In time, the full correlation function, $C_q(0, t)$ is determined, whereas in space such a determination is computationally too demanding for such a large 2D interface (the number of operations grows with the system size to the fourth power). Therefore, we determine the spatial correlation function for the projections of the pile along the x- and y-directions, where along the y-direction the overall slope of the pile is subtracted. For a full two-dimensional determination in space, we have calculated the q-th moment, w_q, of the *width* as well, where the results are consistent with those from the correlation function of the projections (see below).

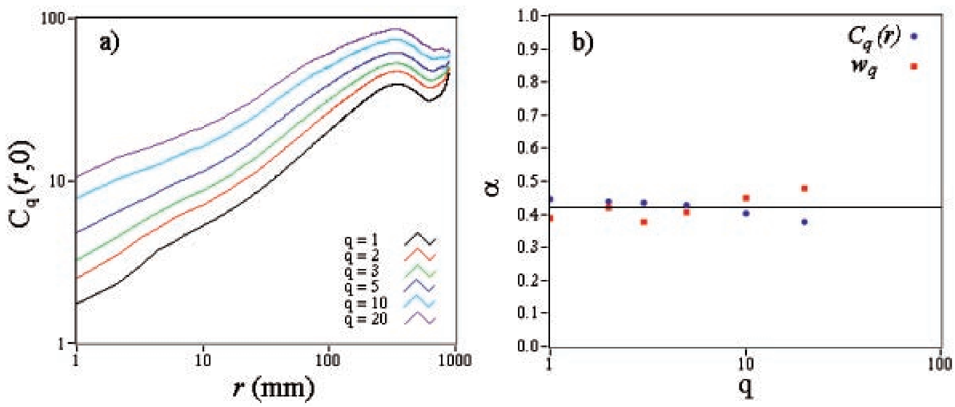

Fig. 2 – (a) Spatial correlation functions of the pile surface, averaged along the x- and y-directions for different moments q. The values of q shown are 1, 2, 3, 5, 10, and 20. Here, the lines shown at increasing height correspond to increasing values of q. In all cases, the slope on a double-logarithmic plot is the same within experimental uncertainty. (b) The dependence of the roughness exponent α on the moment of the correlation function q, from the q-th–order correlation function averaged along the x- and y-directions, as well as the q-th moment of the width in two dimensions. The exponents are determined from a linear fit in the region of 1–30 mm. The values are only slightly dependent on q and are consistent with a constant given the error of the determination. This is in accord with a generic spatial scale invariance of the pile surface, as would be expected for a SOC system.

Results and discussion. – First, we consider the spatial behaviour of the rice pile surface. As an example of the multi-scaling properties, the q-th–order correlation function of the pile, determined in the projections along the x- and y-directions and then averaged, is shown in fig. 2(a) for a range of q values from 1 to 20. As can be seen, there is generic scaling of the pile surface in space, as indicated by the fact that in the double-logarithmic plot of fig. 2(a) all lines are nearly parallel, which means that the roughness exponent is independent of q. This can be seen more clearly in fig. 2(b), where the values of $\alpha(q)$ are shown as a function of q. Here the exponents were obtained from a linear fit to the different order correlation functions on the double-logarithmic scale in the range of 1 to 30 mm. In this figure, we not only show the results corresponding to fig. 2(a), but also from a fully two-dimensional determination of the q-th moments, w_q, of the width instead of the correlation functions, where the exponents have been determined in the same range. As can be seen from the figure, the value of α is consistent with $\alpha = 0.43(5)$ independent of q. This means that the rice pile surface shows generic scale invariance in space, which is what is expected from Paczuski's argument on the multi-scaling behaviour of general SOC systems [5]. This originates from the fact that in the stationary state the surface properties are mainly determined by large avalanches, which means that the presence of a second time scale does not influence the scaling behaviour.

The temporal scaling, however, is radically different. The corresponding q-th–order correlation functions are shown in fig. 3(a), for values of q ranging from 1 to 20. Here, clearly no generic scaling behaviour is observed, but the growth exponent, indicated by the slope of C_q in the double-logarithmic plot, decreases strongly as the moment q increases. This indicates the presence of multi-scaling in the temporal behaviour of the rice pile surface, as predicted by Paczuski for general SOC systems [5]. This is due to the presence of a second time-scale, the scale on which the critical state is reached. This scale also appears in the dynamics of the stationary state and hence leads to multi-scaling. From a simple argument considering the

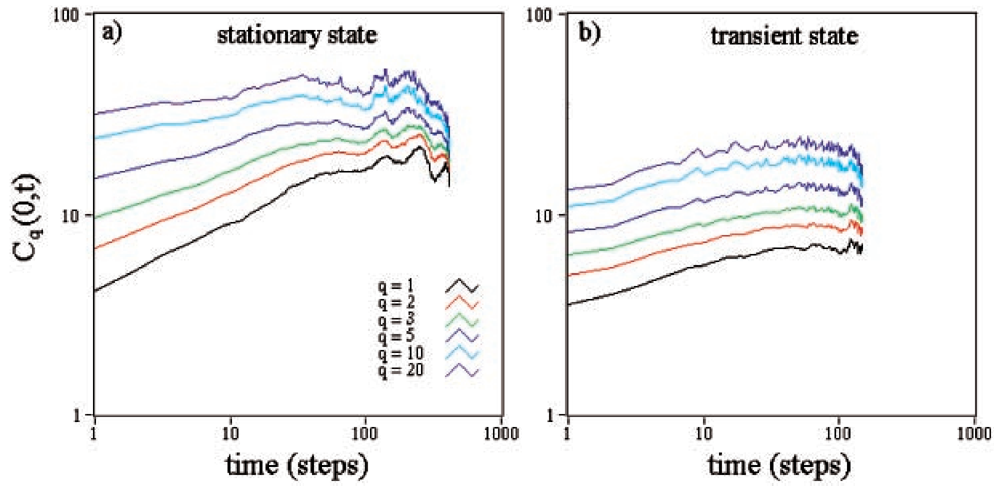

Fig. 3 – Temporal correlation functions of the rice pile for different moments q with $q = 1$, 2, 3, 5, 10, and 20. The lines shown at increasing height correspond to increasing values of q. (a) Results from the stationary state. The slope in the double-logarithmic plot decreases with increasing q, indicating the presence of multi-scaling in time for the rice pile in the stationary state. (b) Results from the transient state. The double-logarithmic plots are parallel for all q, indicating the presence of generic temporal scaling of the rice pile in the transient state.

scaling properties of the q-th moment of the surface, Paczuski predicts a general dependence of the growth exponent β in the stationary state on the moment q as

$$\beta(q) = \beta(q = 1)\frac{D + (q - 1)\alpha}{qD}, \tag{4}$$

where D is the fractal dimension of the avalanches. Note that in ref. [5] the growth exponent for $q = 1$ is defined to be one due to the construction of the model. Therefore, only the q-dependence of the exponent is derived, and effectively time in the model is related to our experimental time to a certain power (given by $\beta(q = 1)$ above).

The fractal dimension of the avalanches in the stationary state of our rice pile has been determined elsewhere [10] to be $D = 1.99(2)$ from finite-size scaling of the avalanche size distributions. Thus with the above determination of the generic roughness exponent, the q-dependence of the growth exponent β can be obtained without any adjustable parameters. In fig. 4, the experimental determination of the q-dependence of β is shown together with the prediction of eq. (4) for both the stationary and the transient states. Here the exponent is determined from a linear fit to the curves in fig. 3(a) over a time scale of 1 to 60 time steps. As can be seen from the figure, the decrease in the observed β is somewhat slower than that predicted by eq. (4) based on SOC behaviour. However, the overall dependence and especially the values of β obtained at high q's are in good agreement with the expectation from SOC, whereas a generic critical system, with a constant value of β, can be ruled out. In the transient state, on the other hand, the figure clearly indicates a value of β independent of the moment q. This is again in good accord with the expectations for a SOC system, where in the transient states only the transient correlation length should be present, hence leading to generic temporal scaling, as is observed in the experiment. The set of correlation functions corresponding to the temporal scaling in the *transient* state is shown in fig. 3(b). Here, it can be seen that all of the double-

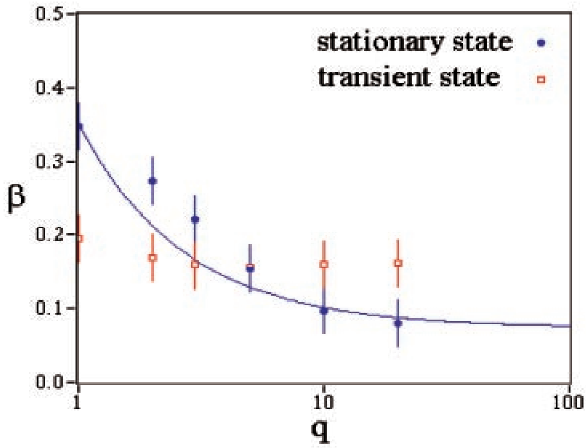

Fig. 4 – The dependence of the growth exponent β on the moment of the correlation function q, for both the stationary and the transient state, determined in the region of 1–60 and 1–30 time steps, respectively. With increasing moment, β decreases in the stationary state, indicating multi-scaling. The solid line indicates the expectation for a SOC system in the stationary state (eq. (4)), which is in reasonable agreement with the experimental results. In the transient state, however, β does not depend on the moment of the correlation function q, indicating generic scaling. Again this is expected for a SOC system due to the absence of the critical correlation length in the transient state.

logarithmic curves are parallel, indicating that the exponents, shown in fig. 4, do not depend on the moment q. Here the exponents are determined from a fit in the range of 1–30 time steps.

Conclusions. – We have presented a multi-scaling analysis of the surface of a three-dimensional pile of rice. In the spatial behaviour, there is generic scaling, as indicated by a roughness exponent independent of the moment q. In the determination of the multi-scaling behaviour, the q-th–order correlation functions were studied in the x- and y-directions, while for a full two-dimensional characterisation, the q-th moments of the width, w_q, were determined. Such generic scaling in space is in good accord with the fact that a rice pile constitutes a SOC system, as in the critical state the surface properties are mainly determined by the occurrence of large avalanches, which leads to a self-affine surface [5].

In time, the behaviour is more complicated, showing multi-scaling in the stationary state, with a decreasing growth exponent as the order q is increased. This is in agreement with the scaling theory of Paczuski [5], where such a dependence of the growth exponent on q is predicted. The reason is that in time there are two fundamental scales, with the self-organization time in addition to the usual dynamical time scale. Experimentally, we find that indeed there is multi-scaling behaviour in the temporal evolution of the surface of the rice pile. The results are also in reasonable agreement with the quantitatively predicted dependence based on SOC (eq. (4)). On the other hand, in the transient state, when the rice pile evolves towards the critical state, there should be only a single time scale present, which leads to generic scaling also in time. This has indeed been observed experimentally, when observing a pile where no system spanning avalanches have yet taken place.

While the observation of finite-size scaling and power law distributed avalanches [10] showed the presence of a critical state in the three-dimensional rice pile, the present observation of multi-scaling in the temporal evolution indicates that this critical state is indeed a self-organized one.

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