Computation of Splitting Functions based on Computer Algebra

Theoretical Particle Physics Group, Prof. Thomas Gehrmann

Thomas Gehrmann, Andreas von Manteuffel, Vasily Sotnikov, and Tong-Zhi Yang*

University of Zurich[UZH]

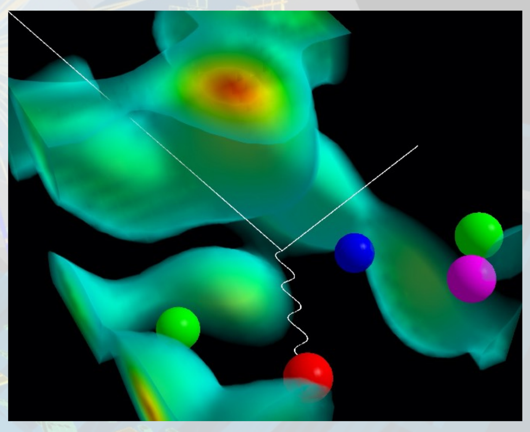
- **QCD factorisation:**

$$\sigma_{h_1 h_2} = \int dx_a dx_b\, f_{a/h_1}(x_a) f_{b/h_2}(x_b)\, \hat{\sigma}_{ab} + \mathcal{O}\left((\Lambda_{\text{QCD}}/Q)^n\right)$$

  - The initial value of Parton Distribution Function $f_{i/h}(x, \mu_0)$ can be fitted from experiment data or computed from lattice QCD.
- **Splitting functions P_{ij} govern the evolution of $f_{i/h}(x, \mu)$**

$$\frac{d f_{a/h}(x,\mu)}{d \ln \mu} = 2 \sum_k \int_x^1 dz\, P_{ak}(z,\mu)\, f_{k/h}(x/z, \mu)$$

- **P_{ij} are fundamental ingredients for collider physics at the LHC.**



See NNLOJET and MATRIX in other two posters for cross section computations.

- **What is computer algebra?**
  - Also called symbolic computation, i.e., manipulates mathematical expressions using a computer.
- **General computer algebra systems**
  - MATHEMATICA, MAPLE, MATLAB, AXIOM …
- **Specific computer algebra systems**
  - FORM, FERMAT, GINAC, GAP …
- **Examples using MATHEMATICA**

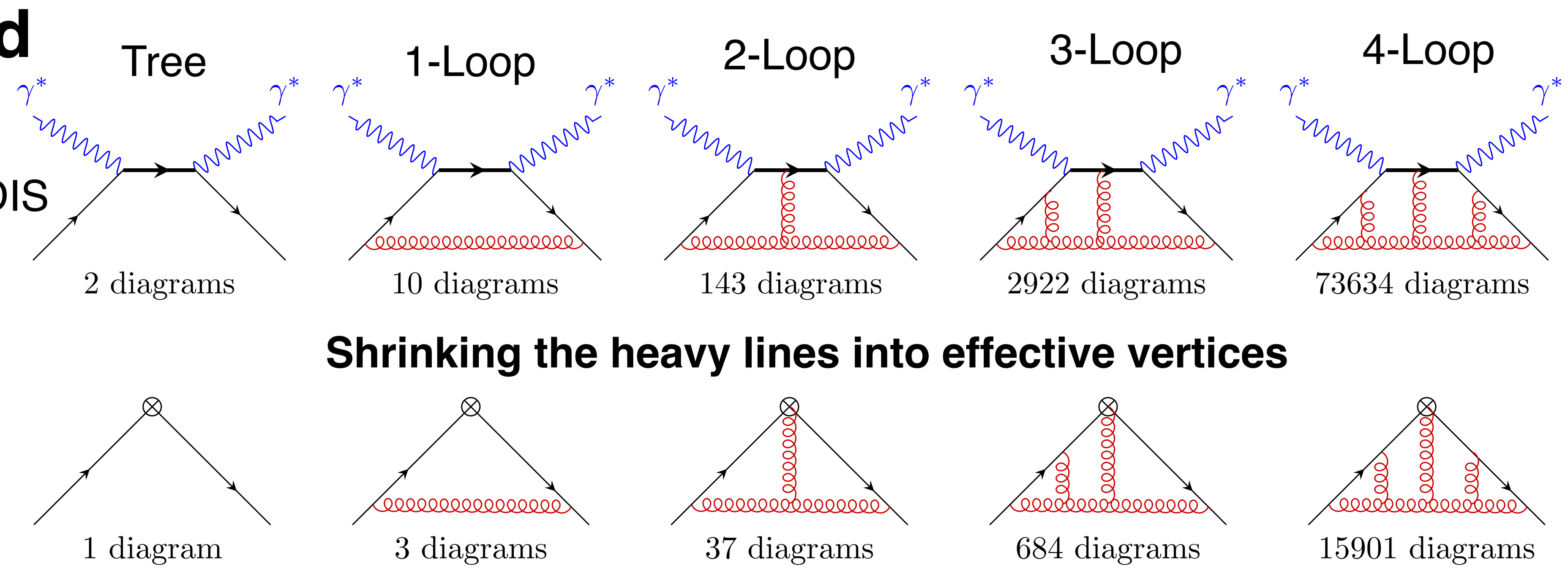
```
In[1]:= Solve[{x + y == 1, x + 2 y == 3}, {x, y}] // Flatten (*solve linear equation system*)
Out[1]= {x → -1, y → 2}

In[2]:= DSolve[D[f[x], x] == f[x]/x, f[x], x] // Flatten (*solve differential equation*)
Out[2]= {f[x] → x C[1]}
```

Method

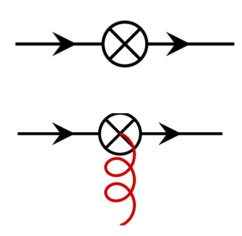
Forward DIS

EFT



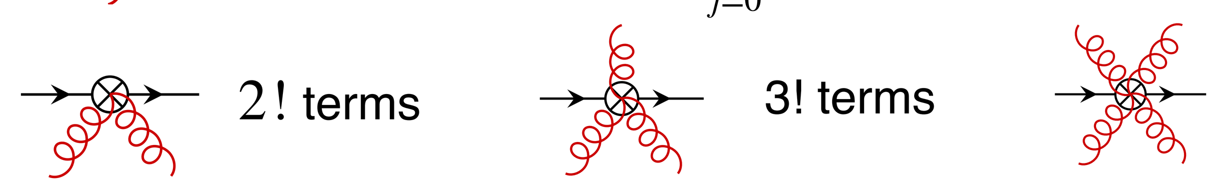
Feynman rules for the effective vertices

The Feynman rules can be obtained by expanding the effective operator $O_q = \bar{\psi}\gamma^{\mu_1} D^{\mu_2} \cdots D^{\mu_N} \psi \Delta_{\mu_1} \cdots \Delta_{\mu_N}$, $D_\mu = \partial_\mu + i g_s T^a A^a_\mu$, $\Delta^2 = 0$



$$\longrightarrow \quad \delta_{ij} \Delta \cdot \gamma (\Delta \cdot p_1)^{N-1}$$

$$\longrightarrow \quad g_s t^a_{ij} \Delta_\mu \Delta \cdot \gamma \sum_{j=0}^{N-2} (\Delta \cdot p_1)^j (-\Delta \cdot p_2)^{N-j-2}$$

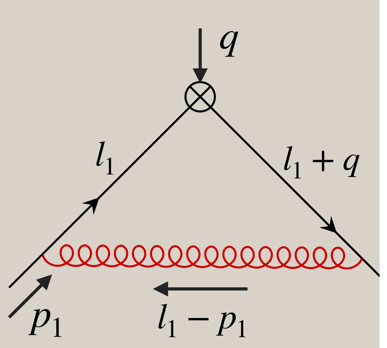


p_1, p_2 are momenta of quark and anti-quark, $k_1, \cdots k_4$ are momenta of gluons, all momenta are incoming.

Trick: $(\Delta \cdot k)^{N-1} \to \sum_{N=1}^{\infty} x^N (\Delta \cdot k)^{N-1} = \frac{x}{1 - x\Delta \cdot k}$, $\sum_{j=0}^{N-2} (\Delta \cdot k_1)^j (\Delta \cdot k_2)^{N-j-2} \to \frac{x^2}{(1 - x\Delta \cdot k_1)(1 - x\Delta \cdot k_2)}$ and so on. Take the coefficient of x^N in the end.

Computational procedure

Step1: QGRAF is used to generate all Feynman diagrams. Its output is symbolic expressions.




```
 diag[3]= (+1)*
pol[quark[-1,p1]]*
pol[phi[-3,q]]*
pol[quark[-2,p]]*
prop[quark[1,l1]]*
prop[gluon[3,l1-p1]]*
prop[quark[5,l1+q]]*
vrtx[qbar[2,-l1],gluon[3,l1-p1],quark[-1,p1]]*
vrtx[qbar[6,-l1-q],phi[-3,q],quark[1,l1]]*
vrtx[qbar[-2,-p],gluon[4,-l1+p1],quark[5,l1+q]];
```

Step2: MATHEMATICA is used to substitute the Feynman rules into the output of QGRAF.

FORM is used to manipulate the Dirac gamma matrix and color algebra, its output is a linear combination of many Feynman integrals classified in topologies.

```
phi1LoopTop[1]=( - 8*Top[1,0,1,1]*cf + 8*Top[1,0,1,1]*eps*cf + 8*Top[1
     ,0,2,0]*cf - 8*Top[1,0,2,0]*eps*cf + 8*Top[1,0,2,1]*cf - 8*Top[1,
     0,2,1]*eps*cf + 8*Top[1,1,1,1]*cf - 8*Top[1,1,1,1]*x*cf - 8*Top[1
     ,1,1,1]*eps*cf + 8*Top[1,1,1,1]*eps*x*cf - 8*Top[1,1,2,0]*cf + 8*
     Top[1,1,2,0]*eps*cf - 8*Top[1,1,2,1]*cf + 8*Top[1,1,2,1]*eps*cf
     - 16/(1 - x)*Top[1,0,1,1]*x*cf + 16/(1 - x)*Top[1,1,1,1]*x*cf);
```

Feynman Integrals

$$\text{Top}[m, a_1, a_2, \cdots, a_n] = \int d^d l_1 \cdots d^d l_j \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}$$

$$d = 4 - 2\epsilon, \epsilon \to 0$$

Example: Top$[1, a_1, a_2, a_3]$

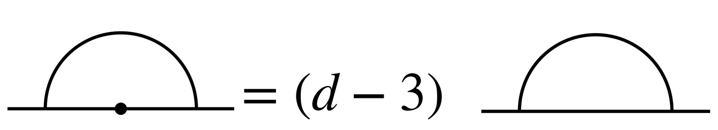
$D_1 = 1 - x\Delta \cdot l_1$, $D_2 = l_1^2$, $D_3 = (l_1 - p)^2$

★Step3: Integration By Parts

- A lot of Feynman integrals need to be evaluated.
- We find relations among Feynman integrals using integration by parts (IBP) and Lorentz invariance.
- This leads to a basis of master integrals.
- Typically,

10^5 Feynman integrals $\xrightarrow{\text{IBP}}$ 10^2 master integrals

- Example:



IBP identity:

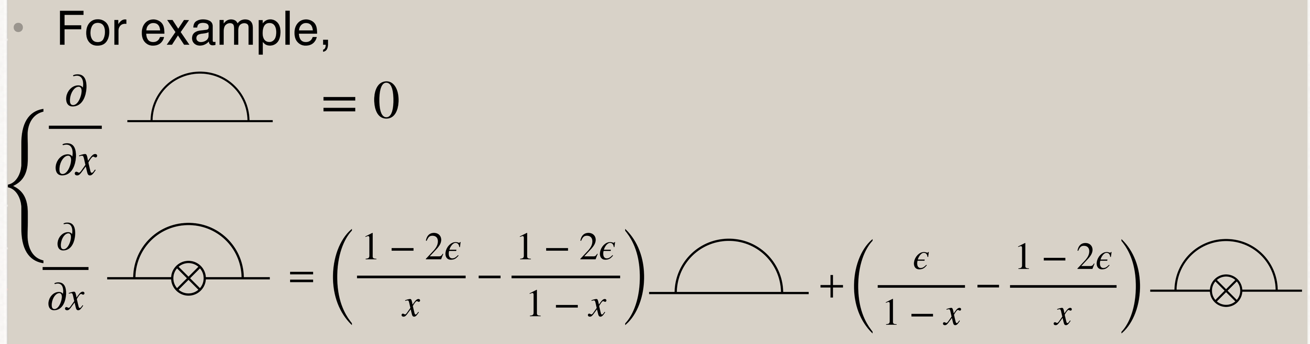
$(a_1 \mathbf{1}^+ - a_3 \mathbf{3}^+ - a_3 \mathbf{2}^- \mathbf{3}^+ - a_1 - 2a_2 - a_3 + d)\text{Top}[1, a_1, a_2, a_3] = 0$

Set $a_1 = 0, a_2 = 1, a_3 = 1$, we get result in the example.

$\mathbf{n}^+$ and $\mathbf{n}^-$ are raising and lowering operators, for example, $\mathbf{3}^+$ Top$[1, a_1, a_2, a_3]$ = Top$[1, a_1, a_2, a_3 + 1]$

★Step4: Differential Equation

- The direct calculation of the master integrals remains to be a challenging task.
- A good method is to differentiate the master integrals with some external variables (x in this computation).
- For example,



- Solving the differential equation (DE) system gives the solutions for the master integrals.
- The solutions are in terms of special functions.

Special Functions

- Classical polylogarithms

$$\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} dt\,, \text{Li}_1(t) = -\ln(1-z)\,.$$

- Harmonic polylogarithms (HPLs)

$$H(a_1, a_2, \cdots, a_n; z) = \int_0^z dt\, f_{a_1}(t)\, H(a_2, \cdots, a_n; t)\,, H(t) = 1\,,$$

$$H(\vec{0}_n; t) = \frac{\ln^n t}{n!}\,, f_1(t) = \frac{1}{1-t}\,, f_0(t) = \frac{1}{t}\,, f_{-1}(t) = \frac{1}{1+t}\,.$$

- Goncharov multiple polylogarithms (GPLs)

$$G(a_1, a_2, \cdots, a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_2, \cdots, a_n; t)\,, G(t) = 1\,,$$

$$G(\vec{0}_n; t) = \ln^n t/n!$$

- Elliptic functions, for example,

$$F(x; m) = \int_0^x dt \frac{1}{\sqrt{(1-t^2)(1-m^2 t^2)}}$$

- More interesting even unknown functions.

Results

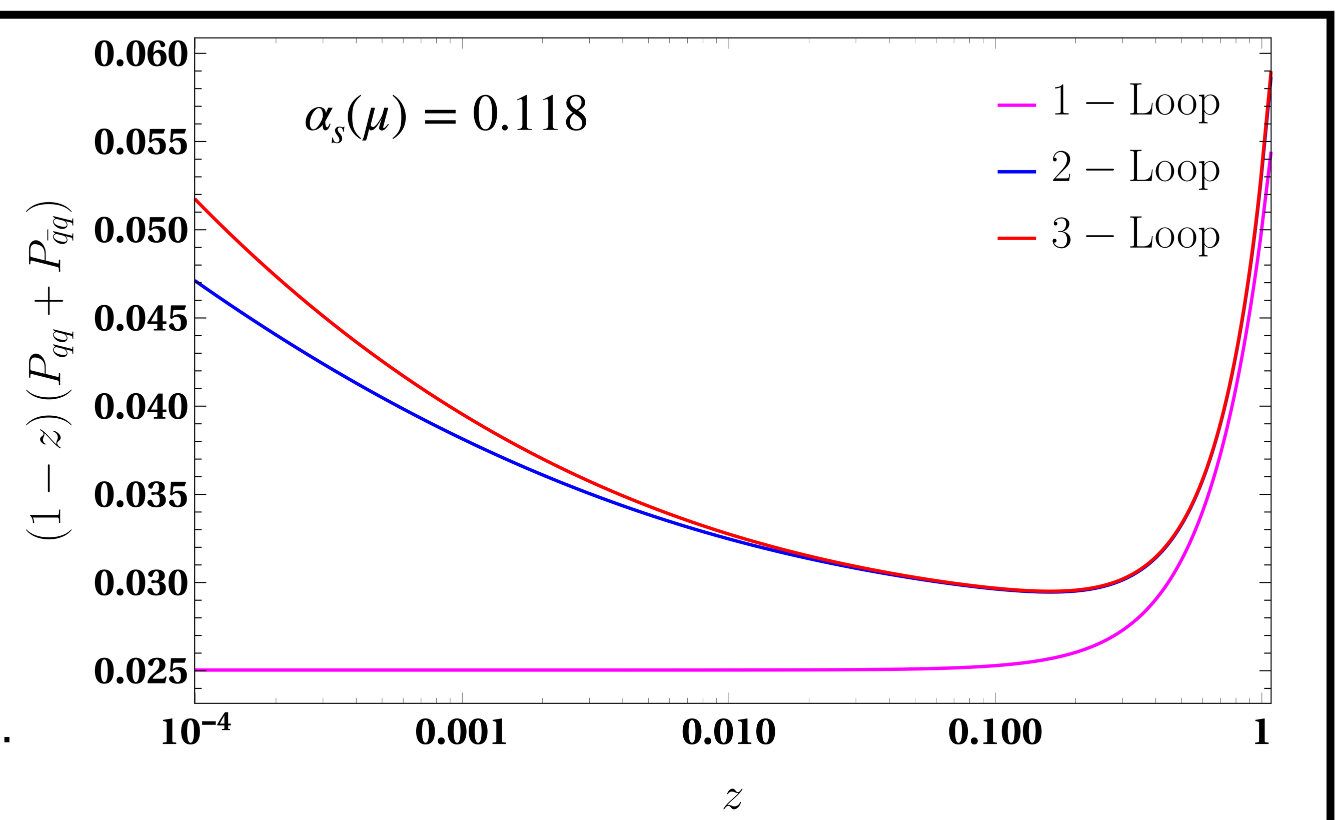
- Take the coefficient of x^N, for example,

$$\frac{1}{1-x} = \sum_{N=0}^{\infty} x^N \to 1\,, \ln(1-x) = \sum_{N=0}^{\infty} x^N \left(\frac{-1}{N}\right) \to \frac{-1}{N}$$

- Perform inverse Mellin transform, Mellin transform is defined as

$$\text{M}_N = -\int_0^1 dz\, z^{N-1} P_{ij}(z)\,.$$

- The results of splitting functions are expressed using HPLs.
- Numerical evaluation of HPLs $\longrightarrow$ Numbers.
- The computation of the four-loop splitting functions is in progress.



*toyang@physik.uzh.ch

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