Gravitational waves and their propagation in different spacetimes

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Abstract

We consider the propagation of gravitational waves (GWs) in a curved background in the presence of the cosmological constant Λ. For this purpose, we present two methods that take different approaches to explain the effects of the cosmological constant on physical quantities. The first method employs a coordinate transformation from the Schwarzschild-de Sitter (SdS) metric to the Friedmann–Lemaître–Robertson–Walker (FLRW) metric. While the SdS metric represents the background spacetime near the source of the GWs, the FLRW metric describes the background of the cosmological observer. With this coordinate transformation, we show that even approximate harmonic waves far away from the source of gravitational radiation could become anharmonic when detected by the cosmological observer. The other method uses a specific coordinate chart, in which the background metric is static and depends on Λ. Using this metric, we show that the amplitude of the wave gets modified with a term dependent on Λ, while the polarizations are the same as before, therefore preserving the quadruple nature. We also calculate the corrections in the geodesic deviation and how it agrees with the accelerated expansion of the universe. In the end, we compare both methods at different orders of Λ.

Contents

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1 Introduction

The cosmological constant Λ has a fascinating history. It was added by Albert Einstein because of his assumption of a quasi-static universe (which was well-founded at the time). However, later it was removed when a cosmological solution (by A. Friedmann) to the original Einstein field equations was able to explain the expansion of the universe implied by Hubble's law. Later physicists again added it to solve the puzzle of the mysterious dark energy and explain other physical phenomena. While the current preferred value of Λ is 1.1×10^{-52}, still we expect deviations in our physical phenomena, however small they are. One such important physical phenomenon are the Gravitational waves (GWs) which were recently directly detected by the LIGO (Laser Interferometer Gravitational-Wave Observatory) detectors for the first time on September 2015 and whose results were published in February 2016 [5]. Due to their decisive contributions to the LIGO's detection of the GWs, three physicists, Rainer Weiss, Kip Thorne, and Barry Barish, received the Physics Nobel prize 2017. While there was some intuition on GWs in the late 19th century due to how much gravity is analogous to electromagnetism, it was Einstein himself who predicted gravitational waves based on the theory of general relativity in his 1916 paper. Its mathematics showed that an accelerating object would cause a disturbance in the spacetime fabric and produce these waves. Some example of the phenomena, which can produce strong GWs are supernovae, the collision of two massive black holes and motion of an imperfect spherically symmetric massive body which is accelerating. Usually, we take the background in which GWs are propagating as flat space because they are far away from their source, but we can also study their behavior for the non-Minkowski backgrounds. Here we review the propagation of GWs in a curved spacetime that includes the cosmological constant Λ.

2 Propagation of GWs in a curved spacetime

The Einstein field equations (EFE) in vacuum (i.e. Energy-momentum tensor $T_{\mu\nu} = 0$) is given by:

$$R_{\mu\nu} - \left(\frac{R}{2} - \Lambda\right) g_{\mu\nu} = 0. \tag{1}$$

Where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is a metric tensor with a signature $(+,-,-,-)$ & Λ is the cosmological constant. We will use geometric units ($G = c = 1$). Expansion of the metric around a background metric is:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}. \qquad |h_{\mu\nu}| \ll |\tilde{g}_{\mu\nu}| \tag{2}$$

Where $\tilde{g}_{\mu\nu}$ is a static background metric and $h_{\mu\nu}$ is a non-static perturbation much smaller than $\tilde{g}_{\mu\nu}$. In this expansion, equation (1) up to first order of h becomes [2]:

$$\tilde{R}_{\mu\nu} + R_{\mu\nu}(h) - \left(\frac{\tilde{R}}{2} + \frac{R(h)}{2} - \Lambda\right) \tilde{g}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) h_{\mu\nu} + \mathcal{O}(h^2) = 0. \tag{3}$$

Here indices are lowered and raised by $\tilde{g}_{\mu\nu}$ up to the first order of h. Components with tilde over it are only dependent on $\tilde{g}_{\mu\nu}$. $R_{\mu\nu}(h)$ and $R(h)$ are given by:

$$R_{\mu\nu}(h) = \frac{1}{2}(h^{\lambda}{}_{\mu;\nu;\lambda} + h^{\lambda}{}_{\nu;\mu;\lambda} - h_{\mu\nu}{}^{;\lambda}{}_{;\lambda} - h^{\lambda}{}_{\lambda;\mu;\nu}). \tag{4}$$

$$R(h) = R^{\lambda}{}_{\lambda}(h) - h^{\lambda\rho}\tilde{R}_{\lambda\rho}. \tag{5}$$

Here ";" denotes the covariant derivative with respect to $\tilde{g}_{\mu\nu}$. We can split equation (3) in two parts, terms independent of h and those which are linearly dependent:

$$\tilde{R}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) \tilde{g}_{\mu\nu} = 0. \tag{6}$$

$$R_{\mu\nu}(h) + \frac{R(h)}{2}\tilde{g}_{\mu\nu} - \left(\frac{\tilde{R}}{2} - \Lambda\right) h_{\mu\nu} = 0. \tag{7}$$

Now we can rewrite equation (7) using equations (4) & (5) by introducing a new quantity $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\tilde{g}_{\mu\nu}$ where $\bar{h}^{\lambda}{}_{\lambda} = \bar{h} = -h^{\lambda}{}_{\lambda} = -h$:

$$\begin{aligned} &\bar{h}_{\mu\nu}{}^{;\lambda}{}_{;\lambda} + \bar{h}_{\lambda\mu}{}^{;\lambda}{}_{;\nu} + \bar{h}_{\lambda\nu}{}^{;\lambda}{}_{;\mu} + 2\tilde{R}_{\lambda\mu\rho\nu}\bar{h}^{\lambda\rho} - \tilde{R}_{\lambda\mu}\bar{h}_{\nu}{}^{\lambda} \\ &\quad - \tilde{R}_{\lambda\nu}\bar{h}_{\mu}{}^{\lambda} - \tilde{R}_{\lambda\rho}\tilde{g}_{\mu\nu}\left(\bar{h}^{\lambda\rho} - \frac{\bar{h}^{\sigma}{}_{\sigma}}{2}\tilde{g}^{\mu\nu}\right) + 2\Lambda\left(\bar{h}_{\mu\nu} - \frac{\bar{h}^{\lambda}{}_{\lambda}}{2}\tilde{g}^{\mu\nu}\right) = 0. \end{aligned} \tag{8}$$

Here it is important to notice that for $\Lambda = 0$ we get $R_{\mu\nu}(h) = 0$ in equation (7). Therefore a non-vanishing Λ adds the last two terms in equation (8). If we impose the Lorentz gauge $\bar{h}_{\mu\nu}{}^{;\nu} = 0$ and traceless condition ($h = 0$, i.e., $\bar{h}_{\mu\nu} = h_{\mu\nu}$) then equation (8) reduces to:

$$h_{\mu\nu}{}^{;\lambda}{}_{;\lambda} + 2\tilde{R}_{\lambda\mu\rho\nu}h^{\lambda\rho} - \tilde{R}_{\lambda\mu}h_{\nu}{}^{\lambda} - \tilde{R}_{\lambda\nu}h_{\mu}{}^{\lambda} - \tilde{R}_{\lambda\rho}\tilde{g}_{\mu\nu}h^{\lambda\rho} + 2\Lambda h_{\mu\nu} = 0. \tag{9}$$

3 Two perspectives on the propagation of GWs in the presence of Λ

Now we present two methods to study the effects of Λ on the GWs. The first method involves doing a coordinate transformation from the Schwarzschild-de Sitter (SdS) metric to the Friedmann–Lemaître–Robertson–Walker (FLRW) metric which represents the backgrounds of the GW source and cosmological observer, respectively [1]. The second method uses a specific coordinate chart in which we can write a static background metric and expand it up to the linear order of Λ. We calculate a correction to the zeroth order of the wave function and also compute the impact of Λ on the geodesic deviation [2].

3.1 Coordinate transformation from SdS to FLRW

The SdS metric in the presence of a static black hole of mass M for a non-vanishing Λ takes the form [6]:

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right) dt^2 - \left(1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{10}$$

Here t is the time coordinate, r is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$ is the angular dependence. Here for $r \to \infty$, it does not reduce to Minkowski metric due to the presence of Λ. Far away from the source of radiation, if our universe is flat space, then GWs are approximately harmonic functions, which can be written as:

$$h^{(GW)}_{\mu\nu}(w; r, t) = \frac{e_{\mu\nu}}{r} \cos[\omega(r - t)]. \tag{11}$$

Here $e_{\mu\nu}$ is the polarization tensor. Due to the expansion of the universe, we are not in Minkowski background space. Instead, an observer far away from the source of GWs will be in a spacetime described by the FLRW [7] metric with coordinates T and R:

$$ds^2 = dT^2 - a(T)^2(dR^2 + R^2 d\Omega^2). \tag{12}$$

Where $a(T)$ is the scale factor. Here $T \neq t$,and $R \neq r$, so equation (11) has different dependence on T & R. Λ introduces some anharmonicity, which should be numerically small as Λ is small ($\approx 10^{-52}$). Conventionally we include the expansion of the universe by redshifting the angular frequency ω in equation (11), i.e. replacing it with:

$$\omega' \approx \omega(1 - z). \; ; \qquad |z| \ll |\omega| \tag{13}$$

Here z is a small redshift factor. For the metric in equation (12), which is spatially flat, we get by using the first Friedmann equation:

$$\frac{\dot{a}(T)}{a(T)} \equiv H = \sqrt{\frac{\Lambda}{3}} \implies a(T) = a_0 e^{\sqrt{\frac{\Lambda}{3}}\Delta T}. \tag{14}$$

Where H is the Hubble constant and the dot is the differentiation with respect to the cosmological time T and $\Delta T = T - T_0$, where T_0 is chosen such that $a(T_0) = a_0$. Here the expansion rate of the universe(i.e., given by H) has a direct dependence on Λ.

In the de Sitter (dS) space we ignore the matter and have a spatially flat universe which is dominated by the cosmological constant Λ. In this space there exist a unique metric, which is completely independent of time, i.e. the SdS metric. The SdS metric in equation (10) for $r \to \infty$ (i.e. far away from a black hole) and $\Lambda \ll 1$ reduces to its linearized form:

$$ds^2 = \left(1 - \frac{\Lambda}{3}r^2\right) dt^2 - \left(1 + \frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{15}$$

This metric is fully time-independent, which implies that the notion of expansion is connected to the coordinates we choose to describe our spacetime. As a cosmological observer detects that the universe is spatially flat and $\dot{a}(T)$ gives the rate at which two massive freely-falling objects separate from each other, we can conclude that the surrounding of the cosmological observer is depicted by FLRW metric (12), as this metric implies a homogeneous and isotropic universe at large scales [8]. The SdS metric approximates the Schwarzschild metric for small r and for large r it gives the de Sitter space and we denote its coordinates as $\{t, r, \theta, \phi\}$. We are interested in the relationship between the $\{t, r\}$ coordinates of the linearized SdS metric (15) and the $\{R, T\}$ coordinates of FLRW. By keeping the spherical symmetry (i.e. $\{\theta, \phi\}$ don't transform), we directly get $r(T, R) = a(T)R$. Hence, we can exactly solve for t (see details in the Appendix) [9]:

$$r(T, R) = a_0 e^{\sqrt{\frac{\Lambda}{3}}\Delta T} R; \quad t(T, R) = T - \sqrt{\frac{3}{\Lambda}} \log \sqrt{1 - \frac{\Lambda}{3} a(T)^2 R^2}. \tag{16}$$

While the SdS metric satisfies a linearized version of the EFE, i.e. equation (3), the FLRW metric does not (as it depends non-analytically on Λ).

Now we expand equations (16) for small values of ΛT^2:

$$\begin{aligned} t(T, R) &\approx T + a_0 \left(\frac{R^2}{2}\sqrt{\frac{\Lambda}{3}} + R^2 \Delta T \frac{\Lambda}{3}\right) + \dots \ . \\ r(T, R) &\approx a_0 \left(1 + \Delta T \sqrt{\frac{\Lambda}{3}} + (\Delta T)^2 \frac{\Lambda}{6}\right) R + \dots \ . \end{aligned} \tag{17}$$

Plugging these in equation (12), we get:

$$ds^2 \approx dT^2 - a_0^2 \left(1 + 2\sqrt{\frac{\Lambda}{3}}\Delta T + \frac{2\Lambda}{3}(\Delta T)^2\right)(dR^2 + R^2 d\Omega^2). \tag{18}$$

Which is also the expansion of equation (12) for small Λ. As $\Lambda \approx 10^{-52}$, we need to consider only the leading order term in $\sqrt{\Lambda}$ for all practical cases.

Taking Minkowski metric as the background (i.e., $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$), it reduces equation (9) to:

$$\Box \tilde{h}_{\mu\nu} + 2\Lambda \eta_{\mu\nu} = 0. \tag{19}$$

Here $\Box = \partial^\mu \partial_\mu$ denotes the d'Alembert operator. We can decompose the $h_{\mu\nu}$ into contributions from the GWs as well as the cosmological constant Λ. Therefore, we can write the metric in equation (2) as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}^{(\Lambda)}_{\mu\nu} + \tilde{h}^{(GW)}_{\mu\nu}. \tag{20}$$

Where wave equation would have the same form i.e.:

$$\Box \tilde{h}^{(GW)}_{\mu\nu} = 0. \tag{21}$$

While the contribution from Λ will take the form:

$$\Box \tilde{h}^{(\Lambda)}_{\mu\nu} = -2\Lambda \eta_{\mu\nu}. \tag{22}$$

Plugging the expression of $t(T,R)$ & $r(T,R)$ from equation (17) into the harmonic waveform $h^{(GW)}_{\mu\nu}(w;r,t)$ in equation (11) transforms $h^{(GW)}_{\mu\nu}(w;r,t)$ to $h^{(FLRW)}_{\mu\nu}(w';R,T)$, which is anharmonic and can be written as:

$$h^{(FLRW)}_{\mu\nu} = h'^{(GW)} \approx \frac{e'_{\mu\nu}}{R}\left(1 + \sqrt{\frac{\Lambda}{3}}T\right) cos\left[\omega(T-R) + \omega\sqrt{\frac{\Lambda}{3}}\left(\frac{R^2}{2} - TR\right)\right]. \tag{23}$$

Here $e'_{\mu\nu}$ is a transformed polarization tensor. Now if we rewrite this equation:

$$h^{(FLRW)}_{\mu\nu} = \frac{e'_{\mu\nu}}{R}\left(1 + \sqrt{\frac{\Lambda}{3}}T\right) cos\left[\omega_{\text{eff}} T - k_{\text{eff}} R\right]. \tag{24}$$

Here $\omega_{\text{eff}} = \omega(1 - R\sqrt{\Lambda/3})$ & $k_{\text{eff}} = \omega(1 - R/2\sqrt{\Lambda/3})$. Now it is important to note that while ω was expected to be redshifted, here k has also transformed. This could be helpful in observing Λ.

In the transverse-traceless (TT) gauge, $e'_{\mu\nu}$ has only non-zero spatial components for X and Y entries (i.e. μ & $\nu = \{1,2\}$). Its temporal components while non-zero are much smaller than the spatial ones so we can ignore them for practical purposes.

In this discussion, we have shown that by including a cosmological constant Λ in our EFE, we are getting a redshift not only in the frequency ω, also change in the wavenumber k. This is a novel effect and will be useful in the observability of Λ.

3.2 Specific coordinate chart

To analyze equation (8) we need to find a suitable background which is far away from the source of gravitational radiation. We know that dS (de Sitter) resp. Ads (Anti-de Sitter) metric solve equation (1) exactly, so it is convenient and natural to choose them as a background. Here we are not taking the SdS metric (10), as we only want to consider a background which is far away from the source. Now we choose a coordinate chart ϕ for the background spacetime $(M, \tilde{g}_{\mu\nu})$, i.e., $\phi : M \to \mathbb{R}^4$, $\phi(m) = (t, x, y, z)$. Let $p : I \subset \mathbb{R} \to M$, be a locus of an observer who is at rest, such that we have $\phi(p(t)) = (t, 0, 0, 0)$. Now we can calculate the solution of equation (6) in the chart ϕ exactly as:

$$\begin{aligned} \tilde{g}_{00} &= \left(\frac{1 - \frac{\Lambda}{12} r^2}{1 + \frac{\Lambda}{12} r^2} \right)^2, \\ \tilde{g}_{ii} &= \frac{-1}{\left(1 + \frac{\Lambda}{12} r^2\right)^2}, \\ \tilde{g}_{ij} &= 0, i \neq j. \end{aligned} \tag{25}$$

It is important to note that the metric in equation (25) is only valid for $r^2 < 12/|\Lambda|$, where $r^2 = 12/|\Lambda|$, i.e. the null horizon, depends upon our choice of p. Also as (25) was introduced in [10], therefore it is called dS resp. AdS for $\Lambda > 0$ resp. $\Lambda < 0$. The Riemann tensor for the metric in equation (25) is:

$$\tilde{R}_{\mu\nu\lambda\rho} = \frac{\Lambda}{3} (\tilde{g}_{\mu\lambda} \tilde{g}_{\nu\rho} - \tilde{g}_{\mu\rho} \tilde{g}_{\nu\lambda}). \tag{26}$$

Here if we impose the Lorentz gauge $\bar{h}_{\mu\nu}{}^{;\nu} = 0$ in equation (8), its trace reduces to $\bar{h}^{;\lambda}{}_{;\lambda} + 2\Lambda \bar{h} = 0$, which can be solved by using the method of separation of variables [11]. However, we are interested in the physical interpretation and consequence of each and every component of $\bar{h}_{\mu\nu}$ due to the inclusion of Λ and not just its trace. Also as we are considering only traceless perturbation then we only get a trivial solution to the trace equation. Therefore we will not use it.

As Λ is small, we only expand equation (8) up to the linear order of Λ. First, we expand our full metric around the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n=1}^{\infty} \Lambda^n h_{\mu\nu}^{(n)}. \tag{27}$$

Here we can also decompose $h_{\mu\nu}^{(n)}$ into $\hat{h}_{\mu\nu}^{(n)} + \check{h}_{\mu\nu}^{(n)}$, where:

$$\sum_{n=1}^{\infty} \Lambda^n \hat{h}_{\mu\nu}^{(n)} = \tilde{g}_{\mu\nu} - \eta_{\mu\nu}, \quad \text{and} \quad \sum_{n=1}^{\infty} \Lambda^n \check{h}_{\mu\nu}^{(n)} = h_{\mu\nu}. \tag{28}$$

Here it is clear that $\hat{h}_{\mu\nu}^{(n)}$ is due to the background metric and $\check{h}_{\mu\nu}^{(n)}$ describes the waveform of the GWs. Also, Λ^n carries the dimension of $[Length]^{-2n}$.

For $r^2 \ll 12/|\Lambda|$, we can expand equation (25) in terms of Λ:

$$\begin{aligned}
\tilde{g}_{00} &= 1 + \mathcal{O}(\Lambda^2) = \eta_{00} + \mathcal{O}(\Lambda^2), \\
\tilde{g}_{ii} &= -1 + \frac{\Lambda r^2}{6} + \mathcal{O}(\Lambda^2) = \eta_{ii} + \mathcal{O}(\Lambda), \\
\implies \tilde{g}_{\mu\nu} &= \eta_{\mu\nu} + \mathcal{O}(\Lambda).
\end{aligned} \tag{29}$$

Which also implies that our Levi-Civita connections, $\tilde{R}_{\lambda\mu\rho\nu}$ & $\tilde{R}_{\mu\nu}$ (due to equation (26)) are of order $\mathcal{O}(\Lambda)$. So we can write equation (8) for traceless perturbation (i.e. $\tilde{h}_{\mu\nu} = h_{\mu\nu}$) as:

$$h_{\mu\nu}{}^{,\lambda}{}_{,\lambda} + h_{\lambda\mu}{}^{,\lambda}{}_{,\nu} + h_{\lambda\nu}{}^{,\lambda}{}_{,\mu} + \Lambda D_{\mu\nu}(h) + \mathcal{O}(\Lambda^2) = 0. \tag{30}$$

Here "," denotes the partial derivative and indices are lowered and raised by $\eta_{\mu\nu}$. Also, $D_{\mu\nu}$ is a linear hyperbolic differential operator of second order. From here on we will neglect terms of the order $\mathcal{O}(\Lambda^2)$.

Equation (30) can be rewritten in the Lorentz gauge ($h_{\mu\nu}{}^{;\nu} = 0$) as:

$$\Box h_{\mu\nu} + \Lambda D_{\mu\nu}(h) = 0. \tag{31}$$

Now we can expand $h_{\mu\nu}$ as a power series of Λ:

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \Lambda h^{(1)}_{\mu\nu} + \mathcal{O}(\Lambda^2). \tag{32}$$

$h^{(1)}_{\mu\nu}$ carries the dimension $[\text{Length}]^2$, therefore we can compare the coefficients order by order in equation (31). Here the zeroth order term simply gives a result in the case of vanishing cosmological constant.

$$\begin{aligned}
\Box h^{(0)}_{\mu\nu} &= 0, \\
\Box h^{(1)}_{\mu\nu} &= -D_{\mu\nu}(h^{(0)}).
\end{aligned} \tag{33}$$

The first part of equation (33) gives plane GWs solution for the "+" and "$\times$" linear polarizations. The second part of equation (33) can be solved exactly as:

$$h^{(1)}_{\mu\nu} = -\mathcal{G} * D_{\mu\nu}(h^{(0)}). \text{ where } \mathcal{G}(t,r) = \frac{\delta(t-r)\theta(t)}{4\pi r}. \tag{34}$$

Here $*$ denotes the convolution and $\mathcal{G}(t,r)$ is the Green's function of the d'Alembert operator. We will integrate only over $\mathbb{R}^3$, which is the lowest order approximation of Ω (i.e. the hypersurface in a coordinate chart ϕ, with $\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid r < 12/|\Lambda|\}$). If we want this integration to converge, we must have $h^{(0)}_{\mu\nu} \to 0$ and by a method of power counting we get $|h^{(0)}_{\mu\nu}| \sim r^{-\alpha}$ for $\alpha > 2$. We also need this set of boundary conditions:

$$\lim_{r\to\infty} h^{(1)}_{\mu\nu} = 0, \quad \text{and} \quad \lim_{r\to\infty} h^{(1)}_{\mu\nu,\lambda} = 0. \tag{35}$$

Still, we will only consider a small region of spacetime (i.e. $r \ll \sqrt{12/|\Lambda|}$), so we do not need to worry about the behavior at $r \to \infty$. Here it is important to note that in this perturbative approach, we assume that $h^{(0)}_{\mu\nu}$ is a Minkowski-plane wave and we restrict the support of it to avoid more loss of symmetry. Hence, we consider a spherically symmetric domain of integration, which can be written as $\Omega_{\mathcal{R}} := \{(x,y,z) \in \mathbb{R}^3 \mid r < \mathcal{R}\}$.

Here we choose a specific coordinate chart ϕ where we only have a plane transverse traceless solution of $h^{(0)}_{\mu\nu}$. In this chart $h^{(0)}_{\mu\nu}$ has only three non-vanishing components, i.e. $h^{(0)}_{11}$, $h^{(0)}_{22} = -h^{(0)}_{11}$ and $h^{(0)}_{12}$. All of them are a function of retarded time $z-t$, so they travel along the z-axis. For this chart, the non-vanishing components of $D_{\mu\nu}(h^{(0)})$ are:

$$
\begin{aligned}
D_{01}(h^{(0)}) &= \frac{7}{6}\left(x\partial_t h^{(0)}_{11} + y\partial_t h^{(0)}_{12}\right), \\
D_{02}(h^{(0)}) &= \frac{7}{6}\left(x\partial_t h^{(0)}_{12} - y\partial_t h^{(0)}_{11}\right), \\
D_{11}(h^{(0)}) &= \frac{r^2}{6}\left(2\partial_{tt} h^{(0)}_{11} - \partial_{zz} h^{(0)}_{11}\right) - \frac{z}{6}\partial_z h^{(0)}_{11} + \frac{2}{3}h^{(0)}_{11}, \\
D_{22}(h^{(0)}) &= -D_{11}(h^{(0)}), \\
D_{12}(h^{(0)}) &= \frac{r^2}{6}\left(2\partial_{tt} h^{(0)}_{12} - \partial_{zz} h^{(0)}_{12}\right) - \frac{z}{6}\partial_z h^{(0)}_{12} + \frac{2}{3}h^{(0)}_{12}, \\
D_{13}(h^{(0)}) &= \frac{5}{6}\left(x\partial_z h^{(0)}_{11} + y\partial_z h^{(0)}_{12}\right), \\
D_{23}(h^{(0)}) &= \frac{5}{6}\left(x\partial_z h^{(0)}_{12} - y\partial_z h^{(0)}_{11}\right).
\end{aligned}
\tag{36}
$$

Now we focus ourselves only on the $''+''$ solutions (analogously we can derive results of $''\times''$). That means we can write $h^{(0)}_{11} = f(z-t) = -h^{(0)}_{22}$ and $h^{(0)}_{12} = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is some function of the retarded time $z-t$. Plugging these in equations (36) yields:

$$
\begin{aligned}
D_{01}(h^{(0)}) &= -\frac{7x}{6}f'(z-t), \\
D_{02}(h^{(0)}) &= \frac{7y}{6}f'(z-t), \\
D_{11}(h^{(0)}) &= \frac{r^2}{6}f''(z-t) - \frac{z}{6}f'(z-t) + \frac{2}{3}f(z-t), \\
D_{22}(h^{(0)}) &= -D_{11}(h^{(0)}), \\
D_{12}(h^{(0)}) &= 0, \\
D_{13}(h^{(0)}) &= \frac{5x}{6}f'(z-t), \\
D_{23}(h^{(0)}) &= -\frac{5y}{6}f'(z-t).
\end{aligned}
\tag{37}
$$

Therefore the first-order correction $h^{(1)}_{\mu\nu}$ can be computed using equation (34):

$$
h^{(1)}_{\mu\nu}(t,\vec{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathcal{R}}} \frac{D_{\mu\nu}(h^{(0)})\left(t - |\vec{x}-\vec{\xi}|, \vec{\xi}\right)}{|\vec{x}-\vec{\xi}|} d^3\xi. \tag{38}
$$

In our coordinate chart ϕ, the only non-zero components are:

$$
\begin{aligned}
h_{11}^{(1)}(t,\vec{x}) = & \frac{1}{24\pi}\int_{\Omega_{\mathcal{R}}}\Bigg[(\vec{\xi})^2 f''\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right) \\
& -\xi_3 f'\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right)+4f\left(\xi_3-(t-|\vec{x}-\vec{\xi}|)\right)\Bigg]\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}, \\
h_{22}^{(1)}(t,\vec{x}) = & \quad -h_{11}^{(1)}(t,\vec{x}).
\end{aligned}
\tag{39}
$$

These results indicate that the polarizations remain the same up to this order which is not the case for the amplitude, thus it is preserving the quadruple nature of gravitational radiation.

Analytically it is extremely hard to solve equations (39), but we can do it for the locus of the observer $p(t)$ using spherical coordinates. If we introduce physical parameters, the frequency ω & the speed of light c and let $h_{11}^{(0)}(t,\vec{0}) = f(\omega t)$, then the non-vanishing components of the perturbation $h_{\mu\nu}$ in equation (2) are:

$$
\begin{aligned}
h_{11}(t,\vec{0}) \approx & \left(h_{11}^{(0)}+\Lambda h_{11}^{(1)}\right)(t,\vec{0}), \\
= & f(\omega t)+\frac{\Lambda}{24\pi}\Bigg[\frac{\mathcal{R}^3\omega}{3c}f'(-\omega t)+\frac{\mathcal{R}^2}{2}\left(f(-\omega t)-f\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right) \\
& -\frac{\mathcal{R}c}{\omega}\left(5f^{\uparrow}(-\omega t)-f^{\uparrow}\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right) \\
& -\frac{2c^2}{\omega^2}\left(f^{\uparrow\uparrow}(-\omega t)-f^{\uparrow\uparrow}\left(\frac{2\mathcal{R}\omega}{c}-\omega t\right)\right)\Bigg], \\
h_{22}(t,\vec{0}) = & -h_{11}(t,\vec{0}).
\end{aligned}
\tag{40}
$$

Here $^{\uparrow}$ denotes a primitive of a function $g : D \subset \mathbb{R} \to \mathbb{R}$ as:

$$
g^{\uparrow}(t) = \int^t g(t')dt'. \tag{41}
$$

We still need to figure out the $\mathcal{R}$, to get the physical understanding of equation (40). A priori $\mathcal{R}$ should be a positive real number, which also gives us the dimension of the support of $h_{\mu\nu}^{(0)}$. While a posteriori for a vanishing Λ in equation (40) these limits should hold:

$$
\lim_{\Lambda\to 0}\frac{\Lambda\mathcal{R}^3\omega}{c} = \lim_{\Lambda\to 0}\Lambda\mathcal{R}^2 = \lim_{\Lambda\to 0}\frac{\Lambda\mathcal{R}c}{\omega} = \lim_{\Lambda\to 0}\frac{\Lambda c^2}{\omega^2} = 0. \tag{42}
$$

It is clear from equation (42) that we have constraints on $\mathcal{R}$ & ω, which are dependent on Λ. We also have another constraint geometric optics limit $\mathcal{R} \gg c/\omega$, which gives us the following inequalities by using equation (42):

$$
\frac{\mathcal{R}^3\omega}{c} \gg \mathcal{R}^2 \gg \frac{\mathcal{R}c}{\omega} \gg \frac{c^2}{\omega^2}. \tag{43}
$$

For our case, we take $\mathcal{R} \ll \sqrt{1/|\Lambda|}$ which satisfies the limits in equation (42) for constant ω. Now we also have $|\Lambda h_{11}^{(0)}| \ll |h_{11}^{(0)}|$, which implies that $\Lambda\mathcal{R}^3\omega/c \ll 1$, which gives us a upper bound on ω. For example, if we take $f(\omega t) = \sin(\omega t)$, then due to the inequalities in equation (43), we can neglect terms with coefficients $\mathcal{R}^2$, $\mathcal{R}c/\omega$ and c^2/ω^2 in equation (40); hence we get:

$$h_{11}(t, \vec{0}) \approx \sin(wt) + \frac{\Lambda \mathcal{R}^3 \omega}{72\pi c} \cos(wt). \tag{44}$$

Furthermore the inequality $\Lambda \mathcal{R}^3 \omega / c \ll 1$ implies:

$$h_{11}(t, \vec{0}) \approx \sin\left(w\left(t + \frac{\Lambda \mathcal{R}^3}{72\pi c}\right)\right). \tag{45}$$

Therefore for a periodic $h^{(0)}_{\mu\nu}$, Λ gives an amplitude modification as per equation (44) and a frequency modification as shown in equation (45).

In the following discussion, we will show that $\mathcal{R}$ depends on the proper time of the observer and hence the frequency also depends on time.

Now we will do our calculations in the lowest order of the chart ϕ, where $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. We consider a source which emits a GWs at some event $(-t_0, \vec{x}_0)$ and which are detected as an approximate plane waves at the event $(0, \vec{0})$ by an observer who is at a large distance $|\vec{x}_0| = t_0$ from the source. Additionally, we also assume that up to the lowest order these waves have a shape of function f. Let the observer at $p(t)$ make a measurement during a time interval $[0, \tau]$, where $\tau \ll t_0$. Then the observer would measure a contribution to f from $h^{(1)}_{\mu\nu}$, which is increasing with τ. These contributions to f originate somewhere in the region $r \leq \tau$, so for present measurement we have $\mathcal{R} = \tau$. Given our conditions, $\tau \ll \sqrt{1/|\Lambda|}$ yields $\mathcal{R} \ll \sqrt{1/|\Lambda|}$. Taking the current preferred value of $\Lambda \approx 10^{-52}$ and the limits on ω as explained before we have:

$$\frac{1}{\tau_{\text{yr}}} 10^{-7}\ \text{Hz} \ll \omega \ll \frac{1}{\tau^3_{\text{yr}}} 10^{15}\ \text{Hz}. \tag{46}$$

Where τ_{yr} is a length of the measurement in years. Equation (46) implies that we have a non-vanishing range for ω. The equation for the geodesic deviation even in a presence of the cosmological constant is the same as for vanishing Λ [3]:

$$\frac{d^2 n^i}{dt^2} = -R^i{}_{00j} n^j. \tag{47}$$

Here $\vec{n} = (n^1, n^2, n^3)$ gives the separation vector between two neighbours of a congruence of timelike geodesics [3]. For the metric in equation (2), we have the Riemann tensor:

$$R_{\mu\nu\lambda\rho} = \tilde{R}_{\mu\nu\lambda\rho} + R_{\mu\nu\lambda\rho}(h) + \mathcal{O}(h^2). \tag{48}$$

Where linear contribution of order h is given by:

$$R^\mu{}_{\nu\lambda\rho}(h) = \frac{1}{2}\left(h^\mu{}_{\nu;\rho;\lambda} + h^\mu{}_{\rho;\nu;\lambda} - h_{\nu\rho}{}^{;\mu}{}_{;\lambda} - h^\mu{}_{\nu;\lambda;\rho} - h^\mu{}_{\lambda;\nu;\rho} + h_{\nu\lambda}{}^{;\mu}{}_{;\rho}\right). \tag{49}$$

Now we configure our detector in such a way that it is only sensitive to the "+" polarization [10], so only non-vanishing components are:

$$\begin{aligned} R^i{}_{00j} &= -\frac{\Lambda}{3}\delta^i{}_j + \mathcal{O}(\Lambda^2), \\ R^1{}_{001}(h) &= -\frac{1}{2}\left(\partial_{tt} h^{(0)}_{11} + \Lambda\left(\partial_{tt} h^{(1)}_{11} + \frac{1}{3}\vec{x}\cdot\nabla h^{(0)}_{11}\right)\right) + \mathcal{O}(\Lambda^2) = -R^2{}_{002}(h). \end{aligned} \tag{50}$$

Plugging these in equation (47), along the locus of $p(t)$, we have:

$$\begin{aligned}
\frac{d^2n^1}{dt^2} &= \left[\frac{1}{2}\frac{d^2h_{11}^{(0)}}{dt^2} + \Lambda\left(\frac{1}{3} + \frac{1}{2}\frac{d^2h_{11}^{(1)}}{dt^2}\right)\right]n^1, \\
\frac{d^2n^2}{dt^2} &= \left[-\frac{1}{2}\frac{d^2h_{11}^{(0)}}{dt^2} + \Lambda\left(\frac{1}{3} - \frac{1}{2}\frac{d^2h_{11}^{(1)}}{dt^2}\right)\right]n^2, \\
\frac{d^2n^3}{dt^2} &= \frac{\Lambda}{3}n^3.
\end{aligned} \tag{51}$$

To find the geodesic deviation as a function of time we decompose $n^i(t)$ into its constant part and its time-dependent small perturbation part, i.e. $n^i(t) = n^i_{(0)} + \delta n^i(t)$, where $|\delta n^i(t)| \ll |n^i_{(0)}|$. Form now on we will write $h_{11}^{(i)}(t) \equiv h_{11}^{(i)}(t,\vec{0})$. Since $h_{11}^{(i)}(0) = \frac{dh_{11}^{(i)}}{dt}(0) = 0$, we get:

$$\begin{aligned}
\frac{n^1(\tau)}{n^1_{(0)}} &\approx 1 + \frac{\delta n^1(0)}{n^1_{(0)}} - \frac{1}{2}h_{11}^{(0)}(0) \\
&\quad + \tau\left(\frac{1}{n^1_{(0)}}\frac{d(\delta n^1)}{dt}(0) - \frac{1}{2}\frac{dh_{11}^{(0)}}{dt}(0)\right) \\
&\quad + \frac{1}{2}h_{11}^{(0)}(\tau) + \Lambda\left(\frac{\tau^2}{6} + \frac{1}{2}h_{11}^{(1)}(\tau)\right), \\
\frac{n^2(\tau)}{n^2_{(0)}} &\approx 1 + \frac{\delta n^2(0)}{n^2_{(0)}} + \frac{1}{2}h_{11}^{(0)}(0) + \\
&\quad \tau\left(\frac{1}{n^2_{(0)}}\frac{d(\delta n^2)}{dt}(0) + \frac{1}{2}\frac{dh_{11}^{(0)}}{dt}(0)\right) \\
&\quad - \frac{1}{2}h_{11}^{(0)}(\tau) + \Lambda\left(\frac{\tau^2}{6} - \frac{1}{2}h_{11}^{(1)}(\tau)\right), \\
\frac{n^3(\tau)}{n^3_{(0)}} &\approx 1 + \frac{\delta n^3(0)}{n^3_{(0)}} + \frac{\tau}{n^3_{(0)}}\frac{d(\delta n^3)}{dt}(0) + \frac{\Lambda\tau^2}{6}.
\end{aligned} \tag{52}$$

Thus contributions from a background gives an isotropic dilation proportional to τ^2, which is due to the expansion of the universe. For $\mathcal{R} = \tau$ in equation (40), the most significant contribution is from the term proportional to τ^3. This term modifies the amplitude of $h_{\mu\nu}^{(0)}(\tau)$ and leads to the loss of periodicity in the zeroes of $\delta n^i(t)$. The effects of the term proportional to τ is similar to the term proportional to the τ^3, while the term proportional to τ^2 only modifies the amplitude. To qualitatively analyze $\delta n^1(t)$, we again introduce the physical units ω and c. Suppose a source starts to emit a GW at an event $(-ct_0, 0, 0, z_0)$ with $|ct_0| < |z|$ and $t_0 \gg \tau$. Also, the source only emits radiation during a time interval of length s. We assume that at the event $(0,0,0,0)$ the observer detects an approximate sine wave up to the lowest order. Thus we have:

$$h_{11}^{(0)}(\omega t) = \varphi(\omega t) := \begin{cases} \sin(\omega t), & 0 \leq t \leq s \\ 0, & \text{otherwise.} \end{cases} \tag{53}$$

We choose the initial conditions:

$$\begin{aligned}\delta n^1(0) &= \frac{n^1_{(0)}}{2} h^{(0)}_{11}(0) \quad \text{and,} \\ \frac{d(\delta n^1)}{dt}(0) &= \frac{n^1_{(0)}}{2} \frac{dh^{(0)}_{11}}{dt}(0).\end{aligned} \tag{54}$$

Hence using equations (40) and (52) and setting $\mathcal{R} = \tau$ & $f = \varphi$, we have:

$$\begin{aligned}\delta n^1(\tau) &\approx \frac{1}{2}\gamma^{(0)}_{11}(\omega\tau) + \Lambda\left(\frac{c^2\tau^2}{6} + \frac{1}{2}\gamma^{(1)}_{11}(\omega\tau)\right) \\ &= \begin{cases} \frac{1}{2}\sin(\omega\tau) + \frac{\Lambda}{24\pi}\left[c^2\tau^2\left(4\pi - \frac{1}{2}\sin(\omega\tau)\right)\right. \\ \left. + \frac{c^2\tau}{\omega} cos(\omega\tau) + \frac{2c^2}{\omega^2}\sin(\omega\tau)\right], & 0 \leq \tau \leq s \\ \frac{\Lambda c^2\tau^2}{6}, & \text{otherwise.} \end{cases}\end{aligned} \tag{55}$$

Here Λ is affecting both the frequency and the amplitude and also adding a term accounting for the isotropic expansion of the universe. If we assume $\Lambda c^2\tau^3\omega \ll 1$ and the geometric optics limit, i.e. $\tau \gg c/\omega$, then we can write the wave-dependent part of a geodesic deviation in equation (55) in a interval $0 \leq \tau \leq s$, in a form:

$$\begin{aligned}\delta n^1(\tau) &= \frac{1}{2}(1 - \delta A_\Lambda)\sin(\omega(\tau + \delta\tau_\Lambda)) \quad \text{where,} \\ \delta A_\Lambda &= \frac{\Lambda c^2\tau^2}{24\pi} \quad \text{and,} \\ \delta\tau_\Lambda &= \frac{2\Lambda c^2\tau}{\omega^2}.\end{aligned} \tag{56}$$

We can deduce from equation (56) that δA_Λ is positive when $\Lambda > 0$, hence decreasing the overall amplitude. This effect might be because of a Λ induced expansion of the universe, and we do expect that an accelerated expansion stretches the GWs.

Here we have shown that for the specific coordinate chart ϕ, we have a static background dS resp. AdS metric (25) (for $\Lambda > 0$ resp. $\Lambda < 0$). Expansion of a background metric (25) for $r \ll \sqrt{12/|\Lambda|}$ and the expansion of $h_{\mu\nu}$ up to the linear order of Λ leads to the general equation (40) for suitable parameter $\mathcal{R}$. We have analyzed equation (40) for periodic function of the form (53) and how the geodesic deviation has modifications in its amplitude as well as in its frequency (55). We also get a term accounting for the isotropic expansion.

4 Analysis

In this project we have discussed two approaches to see the impact of the cosmological constant on physical observables.

In the first method [1], we do a coordinate transformation under appropriate limits, from the linearized SdS metric (15) to the FLRW metric (12). The linearized SdS metric describes a spacetime which is far away from the source of radiation. On the other hand, the FLRW metric represents the background of a cosmological observer which detects the GWs and according to whom the universe is spatially flat and homogeneous at large scale. The approximate harmonic waveform (11), which is expected far away from the source in the case of a flat space background, is not what we found here due to the expansion of the universe. Therefore it introduces some anharmonicity in the wave and a cosmological observer detects the wave of the form as shown in equation (23). Redshift in the frequency (i.e. ω to $\omega_{\text{eff}} = \omega(1 - R\sqrt{\Lambda/3})$ was expected due to the expansion of the universe, while the change in the wave number (i.e. from k to $k_{\text{eff}} = \omega(1 - R/2\sqrt{\Lambda/3}))$ is a novel effect which is due to the cosmological constant.

In the second method [2], we take a suitable coordinate chart in which we get a static background metric (25), namely dS resp. AdS metric (25) (for $\Lambda > 0$ resp. $\Lambda < 0$), which is only valid for $r < \sqrt{12/|\Lambda|}$. For $r \ll \sqrt{12/|\Lambda|}$, equation (8) reduces to (31) for transverse traceless gauge. We can decompose the perturbation in equation (31) up to a leading term of Λ (32) and compare the coefficients order by order (33). Under proper limits, we can solve for $h^{(1)}_{\mu\nu}$ (39), from which we deduce that we are preserving the quadruple nature of the gravitational wave. Including the physical units, frequency ω and speed of light c, we approximately get the full perturbation $h_{\mu\nu}$ (40) for desired polarization ("+" here). Under physical limits (43), we demonstrated that for a periodic function we are getting an amplitude (44) as well as a frequency modification (45). We also computed the effect of cosmological constant on the geodesic deviation (52). For the physical units ω and c, if an observer up to the lowest order starts detecting a sine wave starting from the event $(0, 0, 0, 0)$ for a time interval of length s, then under suitable initial conditions, the geodesic deviation has an amplitude and a frequency modification due to Λ and also an isotropic dilatation which is due to the expansion of the universe (56). In the end, we showed that for a positive Λ, our amplitude of the wave decreases which might be due to the cosmological constant induced expansion, which is expected as an accelerated expansion of the universe stretches the wave (56).

At zeroth order (i.e., $\Lambda = 0$), both methods give a sinusoidal form for the gravitational waves. At first order, it is not possible to compare them as the first method gives the form detected by a cosmological observer as given in equation (24), which is much further from the region of spacetime where we are analyzing the second method. While an approximate coordinate transformation (given in the Appendix) is possible from the metric in equation (25) to the FLRW metric, it is difficult to draw inferences from this transformation. This is due to the integral in equation (39), which is analytically hard to solve for any general event $(t, \vec{x})$ other than along the locus of the observer $p(t)$, i.e. $(t, \vec{0})$. On the other hand, change in the frequency and the wavenumber in equation (24) depends on the non-zero radial coordinate. Therefore, it is analytically hard to compare them at first order.

5 Conclusion

We presented two perspectives to observe the effects of the inclusion of the cosmological constant Λ in the Einstein field equations. Both methods use two vastly different approaches to show the impact of Λ on the physical observables related to gravitational waves. One method employs a coordinate transformation from the SdS metric to the FLRW metric and reveals that there would be a change in the wave number, which is at the root of the observability of Λ. The other uses a specific coordinate chart in which we get a static background metric, which can be expanded up to the linear order of Λ which leads to the modification of the amplitude and the frequency of the geodesic deviation and also adds an isotropic dilatation term to it. Both methods give a sinusoidal form of waves at zeroth order, but at first order it is not possible to compare them as they are propagating in different regions of spacetimes.

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6 Appendix

- **Coordinate transformation from the SdS metric to the FLRW metric for a spherical symmetric spacetime :**

Preserving the spherical symmetry during the transfromation :

$$r(T,R) = a(T)R$$

$$\Longrightarrow dr = R\dot{a}(T)dT + a(T)dR$$

Plugging dr and r in equation (15) :

$$ds^2 = \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)dt^2 - \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}(R\dot{a}(T)dT + a(T)dR)^2$$
$$- (a(T)R)^2 d\Omega^2$$

Which should be equal to (12) :

$$\Longrightarrow dT^2 - a(T)^2 dR^2 = \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)dt^2 - \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}(R\dot{a}(T)dT + a(T)dR)^2$$

$$\Longrightarrow dt^2\left(1 - \frac{\Lambda}{3}(a(T)R)^2\right) = dT^2\left[1 + \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}(\dot{a}(T)R)^2\right]$$
$$- a(T)^2 dR^2\left[1 - \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}\right] + 2dRdT\left[(Ra(T)\dot{a}(T))\left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}\right]$$

Using (14) we can reduce it to

$$\Longrightarrow dt^2 = \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-2}\left[dT^2 + \frac{\Lambda}{3}a^4(T)R^2 dR^2 + 2Ra^2(T)\sqrt{\frac{\Lambda}{3}}dRdT\right]$$

$$\Longrightarrow dt = \left(1 - \frac{\Lambda}{3}(a(T)R)^2\right)^{-1}\left[dT + Ra^2(T)\sqrt{\frac{\Lambda}{3}}dR\right]$$

$$\Longrightarrow t(T,R) = T - \sqrt{\frac{3}{\Lambda}}\log\sqrt{1 - \frac{\Lambda}{3}a(T)^2R^2}.$$

- **Approximate Coordinate transformation from the metric $\tilde{g}_{\mu\nu}$ to the FLRW metric for a spherical symmetric spacetime :**

Preserving the spherical symmetry during the transfromation :

$$\frac{r}{(1+\frac{\Lambda}{12}r^2)} = a(T)R \implies r \approx a(T)R\left(1+\frac{\Lambda}{12}(a(T)R)^2\right)$$

$$\implies dr\left(1-\frac{\Lambda}{4}r^2\right) \approx (R\dot{a}(T)dT + a(T)dR)$$

Plugging dr and r in the spacetime interval of the metric $\tilde{g}_{\mu\nu}$ in the equation (25) :

$$ds^2 \approx \left(1-\frac{\Lambda}{3}\left(a(T)R\Big(1+\frac{\Lambda}{12}r^2\Big)\right)^2\right)dt^2 - \left(1+\frac{\Lambda}{2}r^2\right)\left(1-\frac{\Lambda}{6}r^2\right)(R\dot{a}(T)dT + a(T)dR)^2$$
$$-(a(T)R)^2 d\Omega^2$$

$$\implies ds^2 \approx \left(1-\frac{\Lambda}{3}(a(T)R)^2\right)dt^2 - \left(1+\frac{\Lambda}{3}(a(T)R)^2\right)a(T)^2\left(R\sqrt{\frac{\Lambda}{3}}dT + dR\right)^2$$
$$-(a(T)R)^2 d\Omega^2$$

Which should be equal to (12) :

$$dT^2 - a(T)^2 dR^2 \approx \left(1-\frac{\Lambda}{3}(a(T)R)^2\right)dt^2 - \left(1+\frac{\Lambda}{3}(a(T)R)^2\right)a(T)^2\left(R\sqrt{\frac{\Lambda}{3}}dT + dR\right)^2$$

$$\implies dt^2\left(1-\frac{\Lambda}{3}(a(T)R)^2\right) \approx dT^2\left[1+\frac{\Lambda}{3}(a(T)R)^2\right] - a(T)^2 dR^2\left[1-\left(1+\frac{\Lambda}{3}(a(T)R)^2\right)\right]$$
$$+2dRdT\left[(Ra^2(T)\sqrt{\frac{\Lambda}{3}}\left(1+\frac{\Lambda}{3}(a(T)R)^2\right)\right]$$

$$\implies dt^2 \approx dT^2\left(1+\frac{\Lambda}{3}(a(T)R)^2\right)^2 + \frac{\Lambda}{3}a^4(T)R^2dR^2\left(1+\frac{\Lambda}{3}(a(T)R)^2\right)$$
$$+2Ra^2(T)\sqrt{\frac{\Lambda}{3}}\left(1+\frac{\Lambda}{3}(a(T)R)^2\right)\sqrt{\left(1+\frac{2\Lambda}{3}(a(T)R)^2\right)}dRdT$$

$$\implies dt^2 \approx \left(1+\frac{2\Lambda}{3}(a(T)R)^2\right)dT^2 + \frac{\Lambda}{3}a^4(T)R^2dR^2 + 2Ra^2(T)\sqrt{\frac{\Lambda}{3}}\sqrt{\left(1+\frac{2\Lambda}{3}(a(T)R)^2\right)}dRdT$$

$$\implies dt \approx \left(1+\frac{\Lambda}{3}(a(T)R)^2\right)dT + Ra^2(T)\sqrt{\frac{\Lambda}{3}}dR$$

$$\implies t \approx T + \sqrt{\frac{\Lambda}{3}}a(T)^2R^2$$

Therefore transformation from the $\{t, r\}$ coordinates to the $\{T, R\}$ coordinates is given by :

$$\boxed{\begin{aligned} &r(T,R) \approx a(T)R\left(1+\frac{\Lambda}{12}(a(T)R)^2\right), \\ &t(T,R) \approx T + \sqrt{\frac{\Lambda}{3}}a(T)^2R^2. \end{aligned}}$$