Gravitomagnetic clock effect:
An alternative test of general relativity

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1 Introduction

Einstein's general relativity predicts that a rotating mass like earth, spacetime around which is given by the Kerr metric, affects the motion of test particle moving under its influence in a non-newtonian or purely relativistic way. This effect is usually termed as frame dragging. There are several observable consequences of this effect which has been previously used as an alternative test of general relativity. The two of them are [1], Lens-Thirring effect (which was firstly predicted in 1918 and measured by the LAGEOS mission and will be further improved using the LARES satellite [1]) and the Schiff effect (firstly calculated in 1960, which describes the precession of a gyroscope orbiting a rotating object, and was measured by the Gravity Probe B experiment).
Gravitomagnetism is an interesting analogy to electrodynamics. This effect is a consequence of the rotation of a mass, analogous to magnetism which can originate from a rotating charge. This idea led several works in the direction of Gravitoelectromagnetism [2]. In the quest for understanding Gravitomagnetism, Cohen and Mashoon proposed a third type of observable consequence known as Gravitomagnetic clock effect [3]. In its simplest form this effect can be described as, two clocks that are kept in two orbits around a rotating mass with same orbital parameter but in a pro(with sense of rotation same as that of the gravitating mass)- and a retro(with sense of rotation same as that of the gravitating mass)-grade orbit, the clocks register a time difference. To the lowest order approximation, this time difference is independent of the distance from the rotating mass, the fact which could be a primary enticement for the experimentalists.
Ever since its first prediction in 1993 [3], the theoretical efforts and proposals have been made to analyze the feasibility of its measurements [1]. However, the major milestones in these quests were extending the explanation to different types of orbits to get it closer to the existing experimental resources. The first proposal discussed the ideal circular equatorial orbits, then the analysis was extended to non-equatorial spherical orbits, further to elliptical orbits which are close to the currently available satellite systems. However, in 2014 Lämmerzahl et al [1] made a major breakthrough by proposing the measurement of this effect by comparing the time registered by two clocks in arbitrarily different orbits. From an experimental point of view, this is a very important achievement. It is no longer required to place two satellites in two orbits with same orbital parameter and merely opposite sense of rotation. Such requirements were not practical. However, if two satellites with different orbital parameters can be used for this purpose, it doesn't only make it practically feasible but also removes the requirement for a dedicated mission for this experiment. In principle, the mission equipment can be a piggyback payload on the existing missions and even the data from existing clocks on the satellite can be used.
This report compiles the main theoretical and mathematical developments in this area in a chronological order. Starting from the first proposal to the recent proposal of comparing the time registered by satellites in independent orbits, key mathematical and theoretical approaches proposed are discussed and explained. The primary purpose of this report is educative and it doesn't present any new result. It rather presents all the essential mathematics and the journey of its development in one place. One point worth mentioning here is, different literature written in this context are highly inconsistent with each other in terms of use of notations. It gets very confusing once all these results are getting compiled at one place. Therefore, an effort is made here to bring the consistency in the notations. Towards the end of the report, the error budget and technological feasibility of the proposed experiment is briefly reviewed on the basis of recent technological developments.

2 Theory

The primary idea of general relativity lies in the quote by John Wheeler [4], 'Spacetime tells matter how to move; matter tells spacetime how to curve.' In a typical approach of analyzing the motion of a test mass in the spacetime sourced by a heavy mass, one computes the metric which characterizes the spacetime sourced by the heavy mass. The requirement of one mass being much heavy with respect to the test mass is to justify the assumption that the motion of the test mass doesn't affect the structure of spacetime itself. This makes further calculations relatively simpler. In this situation, where the motion of a test mass in the spacetime sourced by a rotating mass is analyzed, finding the corresponding metric would mark the inaugural step. The metric, derived from a solution to the Einstein field equation, characterizes all the properties of the given spacetime. The spacetime metric of a rotating mass has been worked out by Kerr in 1963 [5] and is a widely popular example of a stationary solution of Einstein field equation.

The full Kerr metric of rotating gravitating body is given by (in Boyer and Lindquist coordinate system [5]),

$$-c^2 d\tau^2 = ds^2 = -\left(1 - \frac{2GMr}{c^2\rho^2}\right) c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2GMra^2}{c^2\rho^2}\sin^2\theta\right)\sin^2\theta \, d\phi^2 - \frac{4GMra\sin^2\theta}{c\rho^2}\, dt\, d\phi \quad , \tag{1}$$

here,

$$\rho^2 = r^2 + a^2\cos^2\theta \quad , \tag{2}$$

$$\Delta = a^2 - \frac{2GMr}{c^2} + r^2 \quad , \tag{3}$$

and the other symbols have their standard meanings (as commonly used in general relativity and spherical coordinate system). Now by finding the corresponding geodesic equations, the consequences of the motion of a test particle in such metric can be predicted. Gravitomagnetic clock effect is one such consequence firstly predicted by Mashoon et al [3]. Further in this report, the Gravitomagnetic clock effect for test particles moving in different types of orbit in the Kerr spacetime is calculated, in a chronological order with respect to the date of their prediction and the level of their difficulties.

2.1 Circular equatorial orbits

To understand the gravitomagnetic clock effect the first example considered is of a test mass orbiting in the equatorial plane of a rotating mass. This situation was analyzed by Mashoon et al in [3], predicting the gravitomagnetic clock effect for the first time. An example can be a satellite orbiting around the earth in an equatorial circular orbit. In such case, one would have $r = r_0 =$ constant and $\theta = \frac{\pi}{2} = constant$, which implies $dr = 0, d\theta = 0$. In this section, for the mathematical simplicity $c = 1$ is assumed in the beginning and it will be substituted back in the final result to maintain the dimensional consistency. These substitutions can simplify the metric mentioned in eq. (1) to a great extent and it reduces to,

$$\begin{aligned} ds^2 &= -dt^2 + (r^2 + a^2)d\phi^2 + \frac{2m}{r}(dt - ad\phi)^2 \quad , \\ &= -\left(1 - \frac{2m}{r}\right) dt^2 + \left(r^2 + a^2 + \frac{2ma^2}{r}\right) d\phi^2 - \frac{4ma}{r} dt d\phi \quad , \end{aligned} \tag{4}$$

where, $m = \frac{GM}{c^2}$. The corresponding Lagrangian can be written as,

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \quad , \tag{5}$$

$$= -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(r^2 + a^2 + \frac{2ma^2}{r}\right)\dot{\phi}^2 - \frac{4ma}{r}\dot{t}\dot{\phi} \quad . \tag{6}$$

Here the dot represents derivative with respect to the proper time. To obtain the corresponding geodesic equation in r, the Euler-Lagrange equation can be used,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad . \tag{7}$$

Because the Lagrangian doesn't explicitly depends on $\dot{r}$, the LHS of the above equation should be zero, thus,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= 0 \quad , \\ \Rightarrow \frac{-2m}{r^2}\dot{t}^2 + \left(2r - \frac{2ma^2}{r^2}\right)\dot{\phi}^2 + \frac{4ma}{r^2}\dot{t}\dot{\phi} &= 0 \quad , \\ \Rightarrow \left(a^2 - \frac{r^3}{m}\right)\dot{\phi}^2 - 2a\dot{t}\dot{\phi} + \dot{t}^2 &= 0 \quad , \\ \Rightarrow \left(a^2 - \frac{r^3}{m}\right) - 2a\frac{\dot{t}}{\dot{\phi}} + \left(\frac{\dot{t}}{\dot{\phi}}\right)^2 &= 0 \quad , \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \left(a^2 - \frac{r^3}{m}\right) - 2a\frac{dt}{d\phi} + \left(\frac{dt}{d\phi}\right)^2 &= 0 \quad , \\
\Rightarrow \frac{dt}{d\phi} &= a \pm \left(\frac{r^3}{m}\right)^{\frac{1}{2}} \quad , \\
&= a \pm \frac{1}{\omega_0} \quad , \\
&= \pm\frac{1}{\omega_0}(1 \pm a\omega_0) \quad , \\
\text{where, } \omega_0 &= \left(\frac{m}{r^3}\right)^{\frac{1}{2}} = \text{keplerian frequency} \quad , \qquad (8)
\end{aligned}
$$

by substituting it in the eq. (4), one can obtain a relation between the proper time and the ϕ coordinate as,

$$
\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{2m}{r} - 1\right)\left(\frac{dt}{d\phi}\right)^2 + \left(r^2 + a^2 + \frac{2ma^2}{r}\right) - \frac{4ma}{r}\frac{dt}{d\phi} \quad , \qquad (9)
$$

$$
\begin{aligned}
-\left(\frac{d\tau}{d\phi}\right)^2 &= \left(\frac{2m}{r} - 1\right)\left(\frac{1}{\omega_0^2}(1 \pm a\omega_0\right) + \left(r^2 + a^2 + \frac{2ma^2}{r}\right) \mp \frac{4ma}{r\omega_0}(1 \pm a\omega_0) \quad , \\
&= 3r^2 - \frac{r^3}{m} \mp \frac{2a}{\omega_0} \quad , \\
\Rightarrow \left(\frac{d\tau}{d\phi}\right)^2 &= \frac{r^3}{m} - 3r^2 \pm \frac{2a}{\omega_0} \quad . \qquad (10)
\end{aligned}
$$

Now, by considering only the positive root of above equation (without loss of generality),

$$
\begin{aligned}
\frac{d\tau}{d\phi} &= \left(\frac{r^3}{m} - 3r^2 \pm \frac{2a}{\omega_0}\right)^{\frac{1}{2}} \quad , \\
&= \left(\frac{1}{\omega_0^2} - \frac{3m}{r\omega_0^2} \pm \frac{2a}{\omega_0}\right)^{\frac{1}{2}} \quad , \\
&= \frac{1}{\omega_0}\left(1 - \frac{3m}{r} \pm 2a\omega_0\right)^{\frac{1}{2}} \quad . \qquad (11)
\end{aligned}
$$

For the given circular equatorial orbit the RHS of the above equation is a constant, thus one can obtain the proper time period to complete one revolution (considering the initial proper time to be zero) by integrating above equation from $\phi = 0$ to $\phi = 2\pi$, i.e.

$$
\tau_{\pm} = \frac{2\pi}{\omega_0}\left(1 - \frac{3m}{r} \pm 2a\omega_0\right)^{\frac{1}{2}} \quad , \qquad (12)
$$

where, τ_+ (τ_-) represents the proper time registered by a clock moving in a pro(retro)-grade orbit for one full revolution. To find the difference between the two (under the first order approximation) one can write it as,

$$
\begin{aligned}
\tau_+^2 - \tau_-^2 &= \frac{4\pi^2}{\omega_0^2}4a\omega_0 \quad , \\
&= 8\pi a T_0 \quad , \qquad (13) \\
\Rightarrow (\tau_+ - \tau_-)(\tau_+ + \tau_-) &= 8\pi a T_0 \quad , \\
\Rightarrow (\tau_+ - \tau_-)2T_0 &\approx 8\pi a T_0 \quad , \qquad (14) \\
\Rightarrow (\tau_+ - \tau_-) &\approx 4\pi a = 4\pi\frac{a}{c} \quad \text{(substituting back } c \text{ for dimensional consistency)}, \qquad (15) \\
\Rightarrow (\tau_+ - \tau_-) &\approx 4\pi\frac{a}{c} = 4\pi\frac{J}{mc^2} \quad . \qquad (16)
\end{aligned}
$$

This shows that the clocks of two satellite orbiting around a rotating massive body in pro- and retro-grade orbits with same orbital parameters would register a time difference of $4\pi\frac{J}{mc^2}$, where J quantifies the angular momentum associated with the spin of the rotating body. The time difference obtained under the first-order approximation is independent of the orbital radius of the satellite.

2.2 General spherical orbits

In order to extend this analysis to a general spherical orbit, that is an orbit with some inclination i with respect to the equatorial plane, it would be required to linearize the Kerr metric under the conditions $a \ll r$ and $\frac{GM}{c^2} \ll r$. The full kerr metric can be written in a matrix form as [6],

$$\begin{bmatrix} -\left(1-\frac{2GMr}{c^2\rho^2}\right) & 0 & 0 & \frac{-2GMra\sin^2\theta}{c^2\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{-2GMra\sin^2\theta}{c^2\rho^2} & 0 & 0 & \left(r^2+a^2+\frac{2GMra^2}{c^2\rho^2}\sin^2\theta\right)\sin^2\theta \end{bmatrix}$$

where,

$$\rho^2 = r^2 + a^2\cos^2\theta \quad , \tag{17}$$

$$\Delta = a^2 - \frac{2GMr}{c^2} + r^2 \quad . \tag{18}$$

Now, under the assumptions of $a \ll r$ and $\frac{GM}{c^2} \ll r$, only linear terms in $\frac{a}{r} \ll 1$ and $\frac{GM}{c^2 r} \ll 1$ are kept and higher order terms neglected. Therefore,

$$\Delta = r^2 \quad , \tag{19}$$

$$\rho^2 = r^2 \quad , \tag{20}$$

$$\frac{\rho^2}{\Delta} = \left(r^2 + a^2\cos^2\theta\right)\left(a^2 - \frac{2GMr}{c^2} + r^2\right)^{-1} \quad , \tag{21}$$

$$= \left(1 + \frac{a^2}{r^2}\cos^2\theta\right)\left(\frac{a^2}{r^2} - \frac{2GM}{rc^2} + 1\right)^{-1} \quad , \tag{22}$$

$$\approx \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \quad , \tag{23}$$

$$\approx 1 + \frac{2GM}{c^2 r} \quad , \tag{24}$$

and the matrix can be re-written as

$$\begin{bmatrix} -\left(1-\frac{2GM}{c^2r}\right) & 0 & 0 & \frac{-2GMa\sin^2\theta}{c^2r} \\ 0 & \left(1+\frac{2GM}{c^2r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ \frac{-2GMa\sin^2\theta}{c^2r} & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

which can be decomposed (for a better visualization) as,

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} + \begin{bmatrix} \frac{2GM}{c^2r} & 0 & 0 & \frac{-2GMa\sin^2\theta}{c^2r} \\ 0 & \frac{2GM}{c^2r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-2GMa\sin^2\theta}{c^2r} & 0 & 0 & 0 \end{bmatrix}$$

Alternatively,

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 + \frac{2GM}{c^2r}\right)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 - \frac{4GMa\sin^2\theta}{c^2r}cdtd\phi \quad . \tag{25}$$

This can also be written in terms of gravitoelectric (Φ) and gravitomagnetic (Ψ) potentials as defined in [7],

$$ds^2 = -(1-2\Phi)c^2dt^2 + (1+2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 4r\Psi\sin^2\theta cdtd\phi \quad , \tag{26}$$

with,

$$\Phi = \frac{GM}{c^2r} \quad , \tag{27}$$

$$\Psi = \frac{GJ}{c^3r^2} = \frac{GMa}{c^2r^2} \quad . \tag{28}$$

2.2.1 Spherical orbits with a small inclination

In the given linearized Kerr metric, if one would like to obtain the equation of motion for a spherical orbit with a small inclination (i.e. a non-equatorial circular orbit making a small angle i with respect to the orbital plane), the following parametrization of the orbit solves the geodesic equations [7],

$$t = \Gamma_0\left(1 - 3\frac{a\omega}{c}\Gamma_0\right)\tau \quad , \tag{29}$$

$$r = r_0 \quad , \tag{30}$$

$$\theta = \frac{\pi}{2} - i\sin\eta + 3i\frac{a\omega}{c}(1 + 2\Phi_0)\omega\tau\cos\eta \quad , \tag{31}$$

$$\phi = \left(1 - \frac{a\omega}{c}\Gamma_0\right)\omega\tau + \phi_0 \quad , \tag{32}$$

where,

$$\Phi_0 = \Phi(r_0) \quad , \tag{33}$$

$$\Gamma_0 = (1 - 3\Phi_0)^{-\frac{1}{2}} \quad . \tag{34}$$

To understand the meaning of the two remaining parameters, it is better to understand their meaning in the unperturbed orbit and then considering them as the perturbed version of the same parameters. Thus, ω is the proper frequency of the unperturbed orbit with

$$\omega = \omega_0\Gamma_0 \quad , \tag{35}$$

and η (for the unperturbed orbit) is the phase angle in the orbital plane measured from the line of nodes [8] and is given by

$$\eta = \omega\tau + \eta_0 \quad . \tag{36}$$

The two constants η_0 and ϕ_0 are given by initial conditions, i.e. by setting $\tau = 0$ in the respective equations. The validity of the linear perturbation theory is under the assumption $\omega_0\tau \ll (\frac{a\omega_0}{c})^{-1}$. It can be observed that in the absence of rotation (i.e. $a = 0$) when the Kerr metric reduces to Schwarzschild metric, the given parametrization represents a circular orbit with orbital plane oscillating about the equatorial plane. In that case, the coordinate time is directly proportional to the proper time, as one would expect.
It can be seen from eq. (31) and (32) that the period of oscillation of ϕ and θ are not the same. ϕ is periodic for $\phi = \phi_0 + 2n\pi$ (by definition) while θ is periodic for $\eta = \eta_0 + 2n\pi$. For,

$$\phi = \phi_0 + 2\pi = \phi_0 + \left(1 - \frac{a\omega}{c}\Gamma_0\right)\omega\tau_+ \text{ (let } \tau_+ \text{ be the proper time for a } 2\pi \text{ rotation in } \phi) \quad , \tag{37}$$

$$\Rightarrow \omega\tau_+ = \frac{2\pi}{\left(1 - \frac{a\omega}{c}\Gamma_0\right)} \quad , \tag{38}$$

$$\Rightarrow \eta_+ = \eta_0 + \frac{2\pi}{\left(1 - \frac{a\omega}{c}\Gamma_0\right)} = \eta_0 + 2\pi \text{ (only if a = 0, thus only for Schwarzschild metric)} \quad . \tag{39}$$

Therefore, the orbit is not spatially closed in general. In such a situation, one has to propose a definition for the closure of the orbit. Here, the proposed definition would be azimuthal closure which defines a close orbit between ϕ_0 to $\phi_0 + 2\pi$. Under this definition, the gravitomagnetic clock effect can be observed directly from eq (38). For pro-grade and retro-grade orbit one can use opposite signs for a. Therefore,

$$\tau_\pm = \frac{2\pi}{\omega\left(1 \mp \frac{a\omega}{c}\Gamma_0\right)} \quad , \tag{40}$$

$$\approx T\left(1 \pm \frac{a\omega}{c}\Gamma_0\right) \text{ (with } T = \frac{2\pi}{\omega}) \quad , \tag{41}$$

$$\Rightarrow \tau_+ - \tau_- = 2T\frac{a\omega}{c}\Gamma_0 = \frac{4\pi a}{c}\Gamma_0 \quad . \tag{42}$$

It can be confusing that this result doesn't converge to the eq. (16) in the special case of $i = 0$. The eq. (16) also uses the linear approximation which is only valid if $\Phi_0 \ll 1$. In the limit of $\Phi_0 \ll 1$ (i.e. $\Gamma_0 \approx 1$), both results agree to each other. One can notice that the result derived above under proposed approximations ($i \ll 1$) is independent of i. This can be understood by the following argument: The definition proposed for orbital closure is only dependent on ϕ not on θ and the ϕ doesn't depend on i in the given parametrization.

2.2.2 General spherical orbits with an arbitrary inclination

In this section, a general orbit with arbitrary inclination i is considered. The orbital equation has been obtained as follow [7] (with the derivation presented in the Ph.D. thesis of the author, direct document not available online),

$$t(\tau) = \Gamma_0 \left(1 - 3\frac{a\omega}{c}\Gamma_0 \Phi_0 \cos i\right)\tau \quad , \tag{43}$$

$$r(\tau) = r_0 \quad , \tag{44}$$

$$\theta(\tau) = \arccos(\sin i \sin \eta) + \frac{3}{2}\frac{a\omega}{c}\Gamma_0(1 - 2\Phi_0)\sin(2i)\frac{\cos\eta}{\sigma(\eta)}\omega\tau \quad , \tag{45}$$

$$\phi(\tau) = \arctan(\cos i \tan \eta) + \frac{a\omega}{c}\left[2\Gamma_0^{-1} + 3\Gamma_0(1 - 2\Phi_0)\frac{\cos^2 i}{\sigma^2(\eta)}\right] \quad , \tag{46}$$

where,

$$\sigma(\eta) = (1 - \sin^2 i \sin^2 \eta)^{\frac{1}{2}} \quad , \tag{47}$$

$$\nu_0 = \phi_0 - \arctan(\cos i \tan \eta_0) \qquad \text{(azimuth of the line of nodes)} \quad . \tag{48}$$

These equations reduce to the previously derived case in the limit of $i \ll 1$. On the other hand, for $i = \frac{\pi}{2}$, the major orbital motion is polar and azimuthal motion can be attributed to the Lens-Thirring precession of nodes [9], thus the gravitomagnetic effect is absent. Therefore, only the case for $i \neq \frac{\pi}{2}$ is analyzed here. To obtain the clock effect, the azimuthal closure is proposed where τ_+ is the proper time registered in a pro-grade orbit to cover $\phi = \phi_0$ to $\phi = \phi_0 + 2\pi$. This can be obtained by solving the implicit eq. (46) perturbatively. Let,

$$\tau_\pm = T \pm 2\pi\frac{a}{c}\lambda \cos i \quad , \tag{49}$$

then for i sufficiently different from $\frac{\pi}{2}$, λ can be evaluated as

$$\lambda = \Gamma_0 - 2\Gamma_0^{-1}\tan^2 i \cos^2 \eta_0 \quad , \tag{50}$$

thus the time difference attributed to the clock effect is given by

$$\tau_+ - \tau_- = 4\pi\frac{a}{c}\lambda \cos i \quad . \tag{51}$$

This result again reduces to the previously derived result for small inclination in the limit $i \to 0$. The gravitomagnetic clock effect should vanish for polar orbits, but that doesn't seem to be a case from eq. (50) and eq. (51). Therefore, it is evident that the eq. (51) is no longer valid when the inclination angle i is sufficiently close to $\frac{\pi}{2}$ ($\tan i \to \infty$). For a spherical polar geodesic orbit, the orbit precesses with the Lense-Thirring frequency $2GJ/c^2a_0^3$, in the same sense as the rotation of the source resulting in the azimuthal closure period of $2\pi(2GJ/c^2a_0^3)^{-1}$. This period is extremely long compared to the orbital period ($\approx 10^{10}$ for near-Earth orbits).Thus, for the validity of the perturbative approach discussed, i must be sufficiently different from $\frac{\pi}{2}$ [10].

2.3 Elliptical orbits

While trying to analyze an elliptical orbit a new strategy has to be used. The analogy between the gravitation and electromagnetism is drawn at this stage and it has been shown that [10] the equation of motion of a particle orbiting in the exterior of a slowly rotating mass, under linear and slow-motion approximation can be given by an equation in Lorentz form [2],

$$\frac{d\mathbf{v}}{dt} = E_g + \mathcal{E}_g + \frac{\mathbf{v}}{c} \times \mathbf{B}_g \quad , \tag{52}$$

Where,

$$E_g = \frac{GM}{r^3}\mathbf{r} \quad \text{(Newtonian gravitoelectric field)} \quad , \tag{53}$$

$$\mathcal{E}_g = \text{Post-Newtonian gravito-electric field} \quad , \tag{54}$$

$$\mathbf{B}_g = \frac{2G}{cr^5}[\mathbf{J}r^2 - 3(\mathbf{J}.\mathbf{r})\mathbf{r}] \qquad \text{(Post-Newtonian gravitomagnetic field)} \quad . \tag{55}$$

As the interest here only lies in the effects caused by the gravitomagnetic correction term, for the following analysis the Post-Newtonian gravitoelectric term $\mathcal{E}_g$ is ignored (it is explicitly calculated in the appendix

of [10]).
The method used to solve this equation, to calculate the gravitomagnetic clock effect, is presented in detail in [10]. Here, only the major strategy used to solve the problem is sketched.
The equation of motion can be written in the form,

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM\mathbf{r}}{r^3} + \frac{2G}{c^2r^5}[3(\mathbf{r}.\mathbf{J})\mathbf{r}\times\mathbf{v} + r^2\mathbf{v}\times\mathbf{J}] \quad . \tag{56}$$

This can be solved by using the perturbation theory sketched here. In the absence of perturbation, the equation would result in an unperturbed elliptical orbit. To solve this equation firstly an inertial coordinate system (X, Y, Z) is proposed. To understand the physical meaning of different parameters mentioned here, it is easier to understand how they manifest in an unperturbed system and then interpreting them as perturbed versions of the same parameters. In an unperturbed system,

$$X = \rho_0 \cos\varphi \quad , \tag{57}$$
$$Y = \rho_0 \sin\varphi \quad , \tag{58}$$
$$Z = 0 \tag{59}$$

where,

$$\rho_0 = \frac{a_0(1-e_0^2)}{1+e_0\cos\hat{v}} \quad , \tag{60}$$
$$\omega_0 t = (\hat{u} - e_0\sin\hat{u}) - (\hat{u} - e_0\sin\hat{u})_{t=0} \quad , \tag{61}$$
$$a_0 = \text{ semimajor axis} \quad , \tag{62}$$
$$e_0 = \text{eccentricity} \quad , \tag{63}$$
$$\hat{v} = \varphi - g_0 = \text{true anomaly of the unperturbed orbit} \quad , \tag{64}$$
$$\hat{u} = \text{eccentric anomaly of the unperturbed orbit} \quad , \tag{65}$$
$$g_0 = \text{argument of the pericenter} \quad , \tag{66}$$
$$\omega_0 = \frac{GM}{a_0^3} = \text{keplerian frequency} \quad , \tag{67}$$
$$T_0 = \frac{2\pi}{\omega_0} \quad , \tag{68}$$

also,

$$\rho_0 = a_0(1 - e_0\cos\hat{u}) \quad , \tag{69}$$
$$a_0(1-e_0^2)\dot{\rho}_0 = e_0 L_0 \sin\hat{v} \quad , \tag{70}$$
$$\text{with, } L_0 = \sqrt{GMa_0(1-e_0^2)} = \text{specific angular momentum of the unperturbed orbit} \quad , \tag{71}$$

where the dot represents differentiation with respect to time t.
In case of a perturbed system the equation of motion in terms of the (X,Y,Z) coordinate (by introducing cylindrical coordinate system (ρ, φ, Z) with $X = \rho\cos\varphi$, $Y = \rho\sin\varphi$) is given by,

$$\ddot{X} + \frac{GMX}{\rho^3} = F_X \quad , \tag{72}$$
$$\ddot{Y} + \frac{GMY}{\rho^3} = F_Y \quad , \tag{73}$$
$$\ddot{Z} + \frac{GMZ}{\rho^3} = F_Z \quad , \tag{74}$$

where F_X, F_Y, F_Z are perturbating accelerations in the respective directions. It turns out to be more convenient to express the equation in the cylindrical coordinate system as follows,

$$\ddot{\rho} - \rho\dot{\varphi}^2 + \frac{GM}{\rho^2} = F_\rho = \frac{\epsilon L_0}{\rho_0^4} \quad , \tag{75}$$
$$\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = F_\varphi = -\frac{\epsilon\dot{\rho}_0}{\rho_0^3} \quad , \tag{76}$$
$$\ddot{Z} + \frac{GM}{\rho^3}Z = F_Z = \frac{2L_0GJ\sin i}{c^2\rho_0^3 a_0(1-e_0^2)}[2(1+e_0\cos\hat{v})\sin\varphi + e_0\sin\hat{v}\cos\varphi] \quad , \tag{77}$$
$$\text{with, } \epsilon = \frac{2GJ\cos i}{c^2} \quad . \tag{78}$$

These equations are solved perturbatively with the detailed solution given in [10]. The orbit is obtained in terms of $X(\varphi), Y(\varphi), Z(\varphi)$.
Furthermore, to define azimuthal closure, the coordinate are transformed in the fundamental frame (x, y, z) where the azimuthal angle can be defined as $\tan\phi = \frac{y}{x}$. To find the time taken to complete a closure, one defines the azimuth ϕ_0 corresponding to $\varphi = \varphi_0$ and $\phi_0 + 2\pi$ corresponds to $\varphi = \varphi_{\mathcal{T}}$. The eq. (76) can be expressed as $\frac{d\rho^2\dot{\varphi}}{dt} = \rho F_\varphi$. This can be integrated to give a function $\dot{\varphi} = f(\rho(\varphi))$ which can further give $\mathcal{T}$ by

$$\mathcal{T} = \int_{\varphi_0}^{\varphi_{\mathcal{T}}} \frac{d\varphi}{f(\rho(\varphi))} \quad . \tag{79}$$

In order to change from pro-grade to retro-grade orbit one needs to change the sign of J. These calculations result in

$$\mathcal{T}_\pm = T \pm 2\pi \frac{J\cos i}{Mc^2}\left[-\frac{3}{1-e^2} + \frac{4 - 2\cos^2\varphi_0 \tan^2 i}{[1 + e\cos(\varphi_0 - g)]^2}\right] \quad . \tag{80}$$

As explained in the previous section, the perturbation theory discussed here is only valid if i is sufficiently different from $\frac{\pi}{2}$. This result, which only takes into account the gravitomagnetic correction, can be written in the form $\mathcal{T}_\pm = T(1 \pm \Xi_{gm})$. In case the Post-Newtonian gravitoelectric term is also taken into account it would take a form of $\mathcal{T}_\pm = T(1 + \Xi_{ge} \pm \Xi_{gm})$. The value of Ξ_{ge} is calculated in the appendix of [10]. Ξ_{ge} is not important here as it would get canceled out when calculating the difference between the periods of the pro- and retro-grade orbits.
However, this result is derived for the coordinate time, the same equation would also hold for the proper time at $\mathcal{O}(c^{-2})$ level, which would be the time registered by spaceborne clocks. Thus, the final result for gravitomagnetic clock effect for an elliptical orbit is given by

$$\tau_+ - \tau_- = 4\pi \frac{J\cos i}{Mc^2}\left[-\frac{3}{1-e^2} + \frac{4 - 2\cos^2\varphi_0 \tan^2 i}{[1 + e\cos(\varphi_0 - g)]^2}\right] \quad . \tag{81}$$

It can be seen that for $e = 0, i = 0$ it reduces to the result derived for a circular equatorial orbit. Also, it is to notice that the result depends on the initial position of the test particles which have to be identical for both clocks. Therefore, the result represents a very special case. The further section would derive a fully general relativistic approach for general orbits in the given Kerr metric.

2.4 Full general relativistic approach for generally independent orbits

In the preceding section, a different approach has been used to understand gravitomagnetic clock effect in an elliptical orbit. The method is different from the approaches used for spherical orbits where directly the geodesic equations obtained from the corresponding metric have been attacked. This section introduces a fully general relativistic approach proposed by Lämmerzahl et al in [1].
In the full Kerr spacetime given by eq. (1), the geodesic equation can be completely separated due to the existence of four constants,

$$\tilde{E} = g_{tt} + g_{t\phi} =: c^2 E = \text{specific energy} \quad , \tag{82}$$

$$\tilde{L_z} = -g_{\phi\phi}\dot{\phi} - g_{t\phi}\dot{t} =: cL_z \quad \text{(specific angular momentum in the direction of symmetry axis)}, \tag{83}$$

$$g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = \epsilon c^2 \quad \text{with } \epsilon = 1 \quad \text{for massive test particle} \quad , \tag{84}$$

$$K = \text{Carter constant}, \qquad \text{such that } K = (aE - L_z)^2 \quad \text{for a motion in the equatorial plane} \; . \tag{85}$$

With these constants of motion, separated equations of motion can be obtained in terms of λ, the Mino-time [11], which is connected to the proper time by $cd\tau = \rho^2 d\lambda$,

$$\left(\frac{dr}{d\lambda}\right)^2 = \mathcal{R}^2 - \Delta(\epsilon r^2 + K) =: R \quad , \tag{86}$$

$$\left(\frac{d\theta}{d\lambda}\right)^2 = K - \epsilon a^2\cos^2\theta - \frac{\mathcal{T}^2}{\sin^2\theta} =: \Theta \quad , \tag{87}$$

$$\frac{d\phi}{d\lambda} = \frac{a}{\Delta}\mathcal{R} - \frac{\mathcal{T}}{sin^2\theta} =: \Phi_r(r) + \Phi_\theta(\theta) \quad , \tag{88}$$

$$c\frac{dt}{d\lambda} = \frac{r^2 + a^2}{\Delta}\mathcal{R} - a\mathcal{T} =: T_r(r) + T_\theta(\theta) \quad , \tag{89}$$

where,

$$\mathcal{R} = (r^2 + a^2)E - aL_z \quad , \tag{90}$$

$$\mathcal{T} = aE\sin^2\theta - L_z \quad , \tag{91}$$

Φ_r and Φ_θ defined here are some functions and are not connected to the previously defined Gravitoelectric potential in section 2.2. The equations can be written in non-dimensional form by using the following substitutions,

$$\bar{r} = \frac{r}{m}, \qquad \bar{t} = \frac{ct}{m}, \qquad \bar{a} = \frac{a}{m},$$
$$\bar{L}_z = \frac{L_z}{m}, \qquad \bar{K} = \frac{K}{m^2}, \qquad \bar{\lambda} = \lambda m \quad . \tag{92}$$

The motion of the test particle doesn't follow a path described by a conic section because of the mismatch of the periodicities of the radial and the latitudinal motions. Therefore, in the following analysis, two different periods of motions are defined. Subsequently, the conjugate fundamental frequencies of the motion are defined. The radial motion oscillates between two values r_p (periapsis) and r_a (apoapsis) and the θ oscillates between θ_{min} to $\theta_{max} = \pi - \theta_{min}$. Then the corresponding periods are given by,

$$\Lambda_r = 2\int_{\bar{r}_p}^{\bar{r}_a} \frac{d\bar{r}}{\sqrt{R(\bar{r})}} \quad , \tag{93}$$

$$\Lambda_\theta = 2\int_{\theta_{min}}^{\theta_{max}} \frac{d\theta}{\sqrt{\Theta(\theta)}} \quad , \tag{94}$$

$$\text{such that,} \quad \bar{r}(\bar{\lambda} + \Lambda_r) = \bar{r}(\bar{\lambda}), \quad \theta(\bar{\lambda} + \Lambda_\theta) = \theta(\bar{\lambda}) \quad , \tag{95}$$

and the conjugate fundamental frequencies given by,

$$\Upsilon_r = \frac{2\pi}{\Lambda_r} \quad , \tag{96}$$

$$\Upsilon_\theta = \frac{2\pi}{\Lambda_\theta} \quad . \tag{97}$$

Considering that the fact the ϕ, t and τ motions are not periodic, the associated fundamental frequencies are defined as follows. The corresponding coordinates are considered to be composed of a part which is linear in λ and the parts which are perturbations in r and θ. Using these ideas one defines,

$$\phi(\bar{\lambda}) = \Upsilon_\phi + \Phi^r_{osc}(\bar{r}) + \Phi^\theta_{osc}(\bar{\theta}) \quad , \tag{98}$$

$$\Upsilon_\phi := \langle \Phi_r(\bar{r}) + \Phi_\theta(\theta) \rangle_{\bar{\lambda}} \quad , \tag{99}$$

$$\text{where,} \quad \langle \cdot \rangle := \lim_{(\bar{\lambda_2} - \bar{\lambda_1}) \to \infty} \frac{1}{\bar{\lambda_2} - \bar{\lambda_1}} \int_{\bar{\lambda_1}}^{\bar{\lambda_2}} \cdot \; d\bar{\lambda} \quad , \tag{100}$$

$$\Phi^r_{osc}(\bar{r}) = \int \Phi_r(\bar{r}) d\bar{\lambda} - \langle \Phi_r(\bar{r})_{\bar{\lambda}} \rangle \bar{\lambda} \quad , \tag{101}$$

$$\Phi^\theta_{osc}(\bar{r}) = \int \Phi_\theta(\bar{r}) d\bar{\lambda} - \langle \Phi_\theta(\bar{r})_{\bar{\lambda}} \rangle \bar{\lambda} \quad , \tag{102}$$

these definitions lead to,

$$\Upsilon_\phi = \frac{2}{\Lambda_r}\int_{\bar{r}_p}^{\bar{r}_a} \frac{\Phi_r(\bar{r})d\bar{r}}{\sqrt{R(\bar{r})}} + \frac{2}{\Lambda_\theta}\int_{\theta_{min}}^{\theta_{max}} \frac{\Phi_\theta(\theta)d\theta}{\sqrt{\Theta(\theta)}} \quad , \tag{103}$$

$$\text{and analogously,} \qquad \Upsilon_t = \frac{2}{\Lambda_r}\int_{\bar{r}_p}^{\bar{r}_a} \frac{T_r(\bar{r})d\bar{r}}{\sqrt{R(\bar{r})}} + \frac{2}{\Lambda_\theta}\int_{\theta_{min}}^{\theta_{max}} \frac{T_\theta(\theta)d\theta}{\sqrt{\Theta(\theta)}} \quad , \tag{104}$$

$$\Upsilon_\tau = \frac{2}{\Lambda_r}\int_{\bar{r}_p}^{\bar{r}_a} \frac{\bar{r}^2 d\bar{r}}{\sqrt{R(\bar{r})}} + \frac{2}{\Lambda_\theta}\int_{\theta_{min}}^{\theta_{max}} \frac{a^2\cos^2\theta d\theta}{\sqrt{\Theta(\theta)}} \quad . \tag{105}$$

Functions $\phi(\bar{\lambda})$, $\bar{t}(\bar{\lambda})$ and $\bar{\tau}(\bar{\lambda})$ have a linear part in $\bar{\lambda}$ and an oscillating terms. Only the linear part contributes in an average secular increase in the coordinates and can be used to define observable quantities like perihelion shift and Lense-Thirring effect. Similarly here, an attempt to define gravitomagnetic clock

effect is made.
Define a function $\bar{\tau} : \phi \to \bar{\tau}(\bar{\lambda}(\phi))$ by using the linearized function $\bar{\lambda}(\phi) = \Upsilon_\phi^{-1}\phi, \quad \bar{\tau}(\bar{\lambda}) = \Upsilon_\tau(\bar{\lambda})$,

$$\bar{\tau}(\phi) := \Upsilon_\tau \Upsilon_\phi^{-1}\phi \quad . \tag{106}$$

This equation gives a general method to calculate the proper time registered by a clock moving in a general orbit in a Kerr spacetime. In order to measure the gravitomagnetic effect, the interest lies in finding the time difference registered by clocks in two different orbits. Unlike previous sections, now any two arbitrary orbits can be compared. To set up an observable comparison, a variable $\bar{\tau}_n(\pm 2\pi; a)$ is defined for two different orbits denoted by $n = 1, 2$. Where a is the Kerr rotation parameter, considered positive here. The sign in front of 2π represents pro- and retro-grade orbits while 2π indicates the full rotation in respective orbits. This variable represents the time registered by the clock while completing an azimuth of $\phi = 2\pi$ in their respective orbits. To extract the purely gravitomagnetic effect, a new observable is defined as,

$$\delta\bar{\tau}_{gm} = \bar{\tau}_1(\pm 2\pi; a) + \alpha\bar{\tau}_2(\pm 2\pi; a) \quad , \tag{107}$$

under the condition that the usual gravitoelectric effects cancel each other, thus for $a = 0$,

$$0 = \bar{\tau}_1(\pm 2\pi; 0) + \alpha\bar{\tau}_2(\pm 2\pi; 0) \quad , \tag{108}$$

$$\to \alpha = -\frac{\bar{\tau}_1(\pm 2\pi; 0)}{\bar{\tau}_2(\pm 2\pi; 0)} \quad , \tag{109}$$

while the sign for 2π should be chosen for each orbit according to their respective sense of rotation. However, the absolute value of $\delta\bar{\tau}_{gm}$ obtained from eq. (107) depends on the numeration of the clocks. It can be seen that,

$$\delta\bar{\tau}_{gm}^{(2,1)} = -\frac{\bar{\tau}_2(\pm 2\pi; 0)}{\bar{\tau}_1(\pm 2\pi; 0)}\delta\bar{\tau}_{gm}^{(2,1)} \quad . \tag{110}$$

In order to remove this ambiguity, one can instead calculate the quantity,

$$\frac{\delta\bar{\tau}_{gm}}{\bar{\tau}_1(\pm 2\pi; 0)} = \frac{\bar{\tau}_1(\pm 2\pi; a)}{\bar{\tau}_1(\pm 2\pi; 0)} - \frac{\bar{\tau}_2(\pm 2\pi; a)}{\bar{\tau}_2(\pm 2\pi; 0)} \quad , \tag{111}$$

the absolute value of which would not change by interchanging the labels of the clocks.
Based on the above discussion, it is clear that one needs to calculate $\bar{\tau}_n(\pm 2\pi; a)$ to be able to compute the corresponding observables in gravitomagnetic clock effect. The basic steps can be sketched as:

1. For both clocks calculate the energies E_n, the angular momenta $L_{z,n}$ and Carter constants K_n ($n = 1, 2$), by using $\frac{dr}{d\lambda}(r_{p,a}) = 0$ in eq. (86) and $\frac{d\theta}{d\lambda}(\theta_{max}) = 0$ in eq. (87).
2. Choose $E > 0$ (without loss of generality), for each orbit. This gives two solutions with L_z positive and negative thus representing pro- and retro-grade orbits. Choose them according to the respective sense of rotation of the orbit.
3. The constants obtained would depend on a, thus find the corresponding Schwarzschild value by setting $a = 0$.
4. Then solve eq. (103) and (105) in terms of elliptic integrals (method explained in appendix of paper [1]) to calculate $\bar{\tau}_n(\pm 2\pi; a)$ and $\bar{\tau}_n(\pm 2\pi; 0)$, which gives the desired observable.

By using a Post-Newtonian expansion scheme explained in [1], the final result is obtained as,

$$\tau(\pm 2\pi, a) \approx 2\pi\sqrt{\frac{d^3}{GM}}\left(1 - \frac{3(1+e^2)m}{2(1-e^2)d}\right) \pm \frac{2\pi(\cos i(3e^2 + 2e + 3) - 2e - 2)}{(1-e^2)^{\frac{3}{2}}}\frac{a}{c} \tag{112}$$

$$\alpha \approx \frac{d_1^{\frac{3}{2}}}{d_2^{\frac{3}{2}}} - \frac{3d_1^{\frac{1}{2}}}{2d_2^{\frac{5}{2}}}\left[\frac{d_1(1+e_2^2)}{1-e_2^2} - \frac{d_2(1+e_1^2)}{1-e_1^2}\right]m \tag{113}$$

$$\delta\tau_{gm} = \frac{a}{c}\left[s_1\frac{2\pi(\cos i_1(3e_1^2 + 2e_1 + 3) - 2e_1 - 2)}{(1-e_1^2)^{\frac{3}{2}}} - s_2\frac{2\pi d_1^{\frac{3}{2}}(\cos i_2(3e_2^2 + 2e_2 + 3) - 2e_2 - 2)}{d_2^{\frac{3}{2}}(1-e_2^2)^{\frac{3}{2}}}\right] \quad . \tag{114}$$

Here, s_1 and s_2 are equal to $+1(-1)$ for pro(retro)-grade orbits. It can be noticed that the expression diverges for $e = 1$, which is not surprising as in such case the orbit itself would diverge (requires $d = \infty$ to have a finite $r_p = d(1-e)$).
Therefore, it can be concluded that by using the approach sketched above, one can calculate the observables corresponding to the clock effect, by comparing the time registered by two clocks in two generally different orbits in a given Kerr spacetime.

3 Experimental perspective

Measuring the gravitomagnetic clock effect by an experiment, thus testing GR with an alternative method, requires two clocks orbiting in a Kerr spacetime. The simplest example would be the clock carried by satellites orbiting in a Kerr-spacetime. A practically feasible experiment can be satellites around the earth, thus measuring the gravitomagnetic clock effect caused by earth's rotation and testing GR at the same time. All approaches comparing the clocks in two satellite orbiting around in a pro- and retro-grade orbit with same parameters requires a dedicated mission to measure gravitomagnetic clock effect. It is not practical and common to have two satellites with same orbital parameters. However, section 2.4 proposes a comparison between two arbitrary orbits. This gives a serious hope for a practically feasible experiment. This doesn't require a dedicated mission and can be performed by a piggyback payload on existing mission satellites. In an ideal experiment, the primary data required from such experiment is the precise orbital parameters and the time registered by the clocks in the satellite. These data should be communicated to the ground stations with required precision to be able to compare the corresponding values of two satellites. Using the precise orbital parameters one can calculate α using eq. (113) and then subsequently the $\delta\tau_{gm}$ using eq. (107) and the time registered by the clocks during the corresponding closure of the orbit. Then the result can be compared to the theoretical value obtained by eq. (114).
In order to understand the precision requirements for such experiments (mainly related to the accuracy of the clock), the corresponding values for existing satellite systems have been calculated in [1]. As a test example, satellites of Global Navigation Satellite System (GNSS) are compared [12]. Most of the GNSS satellites are in a circular pro-grade orbit with different inclinations and carry clocks with a very high frequency-stability. Two examples considered are Galileo and GLONASS, both compared to a geostationary satellite which is included in the Chinese COMPASS system. The respective orbital parameters are as follows,

$$\text{for Galileo,} \quad i_{Ga} = 56^\circ, \quad e = 0, \quad d_{Ga} = 29593\text{km} \quad , \tag{115}$$

$$\text{for GLONASS,} \quad i_{GL} = 64.8^\circ, \quad e = 0, \quad d_{GL} = 25471\text{km} \quad , \tag{116}$$

$$\text{for Geostationary orbit,} \quad i_{Ge} = 0, \quad e = 0, \quad d_{Ge} = 42157\text{km} \quad . \tag{117}$$

The mass and rotation parameters for earth are as follows,

$$m \approx 4.4346 \times 10^{-3}\text{m}, \quad \frac{a}{c} = 1.317 \times 10^{-8}\text{sec} \quad . \tag{118}$$

By putting these values in eq. (114) and (112), one finds for Galileo satellites (compared to geostationary)

$$\frac{\delta\tau_{gm}}{\tau_{GA}(\pm 2\pi; 0)} \approx -1.49 \times 10^{-12} \quad , \tag{119}$$

and that for GLONASS satellites

$$\frac{\delta\tau_{gm}}{\tau_{GL}(\pm 2\pi; 0)} \approx -2.44 \times 10^{-12} \quad . \tag{120}$$

The claimed accuracy of the clock on board of the ESA satellite system [13] is 0.45 nanosec per 12 hours. This results in a fractional accuracy of 1.04×10^{-14}, which is at least 2 order of magnitude better than the required precision. Therefore, in principle with the currently available technology and using existing satellite systems the gravitomagnetic clock effect should be measurable.
However, the presented analysis considers an ideal situation. In case of a practical experiment, several error sources need to be considered. The following section discusses these precision requirements briefly and justifies their feasibility based on recent technological developments.

3.1 The requirements of experimental precisions

An effort is made in [7] to identify major error sources relevant for the proposed experiments. The major error sources have been divided in two groups:

1. Errors due to the tracking of the actual orbit.
2. Deviation from idealized orbit due to a reason of
   - Gravitational origin
     - mass multipole moment of earth.

  - the gravitational influence of the Moon, the Sun, and other planets.
- Non- gravitational origin
  - radiation pressure.
  - other systematic errors (e.g. atmospheric disturbances, solar wind etc.).

For the precision requirements related to the tracking of orbits, it is claimed in [1] that the semimajor axis should be known with an accuracy of about 10 μm, while the inclination should be known to an accuracy of 0.03°. The requirement of the determination of semimajor axis can be relaxed by taking observations over a long period of time as the gravitomagnetic effect accumulates over every revolution. Therefore, within few years of observation (in case of geostationary satellites), this requirement can be easily moved up to the centimeter range, which should be achievable by current technology. Recently proposed orbit prediction techniques in [14] seem promising to achieve these requirements. Furthermore, a technique of using two clocks is also proposed in [1], which can be another easier alternative to bypass this precision requirement.
For the second type of error sources, in [7] it is estimated that the perturbing forces should be kept below 10^{-11} g. However, the mass multipole moment of earth is well above this limit (10^{-3}). Therefore, it would be required to correct the obtained result for this effect up to sufficient precision. The earth's gravity has been already mapped to the required order by the experiments like GRACE and ESA's GOCE [15]. The very recent proposal of mapping earth's gravity by atomic clock claims to push the measurement accuracy to the range of 10^{-17}, which is well above the requirements mentioned here. On the other hand, as mentioned in [7], the position of Moon and Sun are known to much higher accuracy to determine their perturbing effect (which is in order of 10^{-7} g and 10^{-8} g, respectively) and the perturbing effect other planets are below the 10^{-11} gboundary.
The radiation pressure of the Sun causes perturbative accelerations of the order of 10^{-8} g, which is well above the 10^{-11} g boundary. To deal with the such deviations caused by a non-gravitational origin, the technique of drag-free satellite can be employed. As an example, the proposed drag free requirement for LISA Pathfinder, which concluded it's mission in July 2017 [16], was 3×10^{-14} m/s^2 per $\sqrt{Hz}$ (Original article [17], updated preprint [18]). This is well below the requirements proposed here. Therefore, it would be valid to claim that such disturbances can be easily compensated by a currently available drag-free technology.

4 Conclusion and Outlook

The major mathematical developments from the first proposal of the gravitomagnetic effect in 1993 to the present time are discussed in this report. The developments, which were necessary to propose a practically feasible experiment to measure this effect, thus using it as an alternative test for general relativity, are elaborated. With the recent proposal of measuring this effect by comparing data from two individual satellites, it can be claimed that the science case of the proposed experiment is well developed and indicates a feasible experiment to measure the gravitomagnetic clock effect. The major error sources are briefly discussed in this report. It can be concluded that all the major error sources can be either eliminated or compensated by the currently available technologies.
However, the present budgeting of the error sources is done on the order of magnitude level. A detailed analysis of the error budget is required in future and the technology case for the project needs to be developed. Further, there are two prospective experiments that can be proposed. One being a piggyback payload dedicated to this experiment in a forthcoming mission and another by directly using the data from existing missions. Both possibilities should be analyzed and compared in detail. Based on the results of the comparison, a detailed proposal for the experiment then needs to be proposed.
The report compiles the different developments proposed in this field over the period of the 25 years. However, the error budgets are briefly reviewed and their feasibilities are checked on the basis of information available on the recent technologies used in current space missions. This can be taken as a starting point to work on the technology case of the proposed experiment.

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