Hawking Radiation

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Contents

Abstract

This work is a review of some articles and books concerning the phenomenon of Hawking radiation. We will first give a short introduction on the Schwarzschild solution and on Penrose diagrams, which will be useful throughout the whole work. After that, we will discuss quantum field theory on a curved spacetime and we will apply it to the massless Klein-Gordon field on the Schwarzschild spacetime. This will allow us to study the phenomenon of particle creation by a black hole and in particular to calculate the expected number of particles detected far away from it.

1 Introduction

The purpose of this work is to review some literature regarding Hawking radiation. The name Hawking radiation indicates the phenomenon of particle creation by black holes, which arises when considering quantum field theory in a Schwarzschild or Kerr spacetime. In this work, we will focus on studying the production of massless Klein-Gordon particles by a Schwarzschild black hole.

In quantum field theory on a curved spacetime, the metric is treated classically but is coupled to matter fields which are treated quantum mechanically, like in the well known case of quantum field theory in an external classical potential. If one considers a spacetime which has an initial flat region (1), followed by a region of curvature (2) and by a final flat region (3), this theory can be treated like a scattering theory.

In the particular case of the Klein-Gordon field, the deep reason for the creation of particles is contained in the fact that, as we will see, one can express the operator ϕ obeying the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi = 0$ as

$$\phi = \sum_i \left(F_i a_i + \bar{F}_i a_i^\dagger \right) ,$$

where the $\{F_i\}$ are a complete orthonormal family of complex valued positive frequency solutions of the equation $(\partial_\mu \partial^\mu + m^2)F = 0$, while the $\{\bar{F}_i\}$ are their complex conjugates (which represent negative frequency solutions). Indeed, the scattering of these solutions by the classical metric will cause a mixing between the sets $\{F_i\}$ and $\{\bar{F}_i\}$, which results in a different splitting of the operator ϕ into annihilation and creation operator. This means that the initial vacuum state $\Psi_{0(1)}$, i.e. the state satisfying $a_{i(1)}\Psi_{0(1)} = 0$ in region (1), will not be the same as the final vacuum state, because in region (3) we have $a_{i(3)}\Psi_{0(1)} \neq 0$, due to the fact that $a_{i(1)}$ and $a_{i(3)}$ are different operators. This also implies that, in order to have a well defined notion of particle, one has to be able to define the notion of positive and negative frequency unambiguously. As we will see, this is possible in flat spacetime but not, for example, on the black hole horizon. Therefore, it is not clear what the physical meaning of a "particle at the horizon" is.

Following the work of Wald ([2]), we will develop a quantum scattering theory which will allow us to quantify the particle production by a Schwarzschild black hole. In particular, we are interested in calculating the expected number of particles produced by it.

2 Basic notions on Schwarzschild spacetime and Penrose diagrams

In this work, we will study the phenomenon of Hawking radiation created by a black hole formed by a spherically symmetric collapse, i.e. a non-rotating Schwarzschild black hole.

2.1 Schwarzschild solution

The Schwarzschild metric is the most general solution to the Einstein field equations for a spherically symmetric matter distribution of total mass M, under the assumption of no rotation and that the cosmological constant is zero. It is valid in vacuum, i.e., in the space region outside of the mass distribution, and it is static. In units with $c = G = \hbar = 1$, the *Schwarzschild metric* for mass M is defined as

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where t is the time coordinate (measured by a stationary clock located infinitely far from the massive body), r is the radial coordinate (measured as the circumference, divided by 2π, of a sphere centered around the massive body) and θ and φ are the angular coordinates. The radius $r_s = 2M$ is called Schwarzschild radius and defines the event horizon of the black hole. At $r_s = 2M$, the solution seems to have a singularity. This singularity is not physical, but it is due to a bad choice of coordinates.

We can apply the following coordinate transformation

$$T = \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \sinh\left(\frac{t}{4M}\right) \tag{2}$$

$$X = \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \cosh\left(\frac{t}{4M}\right) \tag{3}$$

for $r > 2M$ and

$$T = \left(1 - \frac{r}{2M}\right)^{1/2} \exp\left(\frac{r}{4M}\right) \cosh\left(\frac{t}{4M}\right) \tag{4}$$

$$X = \left(1 - \frac{r}{2M}\right)^{1/2} \exp\left(\frac{r}{4M}\right) \sinh\left(\frac{t}{4M}\right) \tag{5}$$

for $r < 2M$. The new coordinates are called *Kruskal-Szekeres coordinates*. In these coordinates, the Schwarzschild metric takes the form

$$ds^2 = \frac{2M}{r} \exp\left(1 - \frac{r}{2M}\right) (dT^2 - dX^2) - r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{6}$$

One notices that the singularity at $r = 2M$ disappears. Furthermore, it can be easily observed from (6) that the radial light-like geodesics (the worldlines of light rays moving in a radial direction) correspond to diagonal straight lines in the chart. In Kruskal-Szekeres coordinates, the Schwarzschild solution can be extended to regions that are not covered by Schwarzschild coordinates. Specifically, one can show that (6) is a valid solution to the Einstein field equations in vacuum for any $(X, T) \in \mathbb{R}^2$. One distinguishes four regions, defined below and illustrated in figure 1.

Region I	exterior region	$r(T, X) > 2M$ and $X > 0$
Region II	interior of black hole	$0 < r(T, X) < 2M$ and $T > 0$
Region III	parallel exterior region	$r(T, X) > 2M$ and $X < 0$
Region IV	interior of white hole	$0 < r(T, X) < 2M$ and $T < 0$

For the purpose of this work, it will be useful to introduce three more coordinate systems in region I of the Schwarzschild spacetime. The first one is given by *Regge-Wheeler coordinates* $(t, r_*, \theta, \varphi)$, where

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{7}$$

It maps $r \in (2M, \infty) \mapsto r_* \in (-\infty, \infty)$. The metric in these coordinates reads

$$ds^2 = \left(1 - \frac{2M}{r}\right) (dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{8}$$

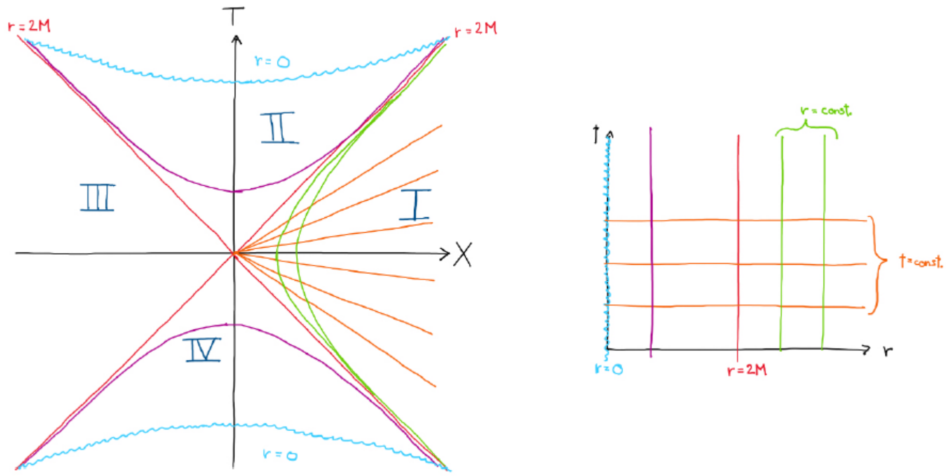

Figure 1: Kruskal-Szekeres coordinates. This picture is taken from [8].

with $r = r(r_*)$. The second system is represented by the *ingoing Eddington-Finkelstein coordinates* (v, r, θ, φ), where $v = t + r_*$ is called *advanced time*. The metric in these coordinates reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{9}$$

The advanced time v is a constant parameter along null geodesics that enter the black hole. The third system is represented by the *outgoing Eddington-Finkelstein coordinates* (u, r, θ, φ), the coordinate $u = t - r_*$ is called *retarded time*. The metric in these coordinates reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{10}$$

The retarded time u is a constant parameter along null geodesics that exit the white hole. One can show (see chapter 6.4 of [4]) that the Eddington-Finkelstein coordinates are related to the Kruskal coordinates (if we set $U = T - X$ and $V = T + X$) in the following way

$$U = -\exp(-\kappa u), \tag{11}$$
$$V = \exp(\kappa v), \tag{12}$$

where U is called *Kruskal retarded time* and V is called *Kruskal advanced time*.

2.2 Penrose diagrams

Penrose diagrams are useful to illustrate the causal relation in a spacetime. They are finite two-dimensional representations of a spacetime with the property that worldlines of massless particles are straight diagonal lines. They are obtained by compactifying light-like coordinates to map them onto a finite interval. The construction of Penrose diagrams is illustrated in section 4.5.3 of [8]. Throughout this work, we will deal with a Schwarzschild spacetime parametrized by Kruskal-Szekeres coordinates. Figure 2 shows the Penrose diagram of the analytically extended Schwarzschild solution. In this diagram, null geodesics in the $X - T$ plane are at $\pm 45°$ to the vertical. Each point of the diagram represents a 2-sphere of area $4\pi X^2$. The following table summarises some particularly interesting points in this diagram.

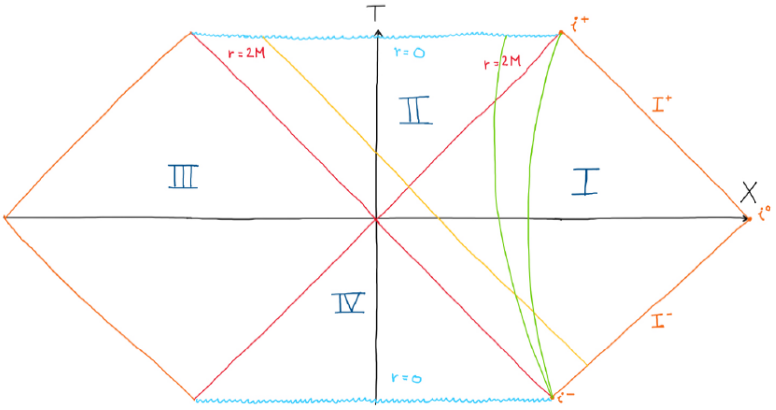

Figure 2: Penrose diagram of the extended Schwarzschild solution. This picture is taken from [8].

i^0	$(T, X) = (0, 1)$	space infinity
i^+	$(T, X) = (1, 0)$	infinite future
i^-	$(T, X) = (-1, 0)$	infinite past
$\mathscr{I}^+$	$T + X = 1$	infinite light-like future
$\mathscr{I}^-$	$T - X = -1$	infinite light-like past.

Most of the diagram is not relevant to a black hole formed by gravitational collapse. The Penrose diagram describing this latter case is depicted in figure 3.

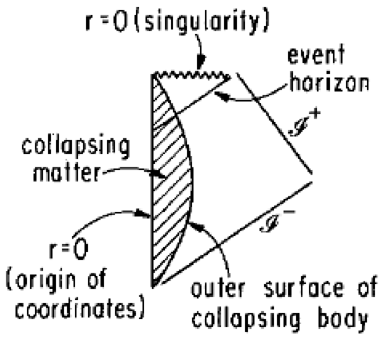

Figure 3: Penrose diagram describing gravitational collapse. This picture is taken from [2].

In this case, the metric is exactly the Schwarzschild metric everywhere outside the surface of the collapsing object, which is represented by a time-like geodesic. Inside the object the metric is completely different, the white hole horizon, the past $r = 0$ singularity and the parallel exterior region do not exist and are replaced by a time-like curve representing the origin of polar coordinates.

2.3 Killing vector field

A Killing vector field K is a vector field whose flow is an isometry

$$\mathcal{L}_K g_{\mu\nu} = 0, \tag{13}$$

where $\mathcal{L}_K g_{\mu\nu}$ denotes the Lie derivative of the metric $g_{\mu\nu}$ with respect to K. A spacetime is called stationary if it admits a Killing vector field that is asymptotically time-like.

The Schwarzschild solution is static (and hence stationary). Therefore, it admits such a Killing field, which we call *time translation Killing vector field*. Moreover, in the presence of a black hole, the Killing vector field is light-like on the event horizon and space-like in the interior of the black hole.

Given Killing vector field K, one can define the surface gravity κ as follows

$$K^a \nabla_a K^b = \kappa K^b. \tag{14}$$

If we consider K to be the time translation Killing vector field of the Schwarzschild solution with mass M, we obtain $\kappa = \frac{1}{4M}$.

3 Some notions of quantum field theory

3.1 Quantum field theory of the real scalar field in flat spacetime

The purpose of this section is to briefly recapitulate the standard quantum field theory of the free Klein-Gordon field in order to establish the notation for the following sections. We will follow section II of [2].

The *quantized Klein-Gordon field* ϕ is a Hermitian operator satisfying the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = 0. \tag{15}$$

The Hilbert space on which the operator ϕ acts is taken to be $\mathscr{H} = L^2(M_+)$ where M_+ is the positive mass shell (i.e. M_+ is the submanifold of the Fourier transformed Minkowski space defined by $-k^\mu k_\mu + m^2 = 0$ with k^μ future directed). This space is interpreted as the Hilbert space of one particle states. The dual Hilbert space to $\mathscr{H}$ is denoted by $\bar{\mathscr{H}}$ and the Hilbert space of states is taken to be the symmetric Fock space $\mathscr{F}(\mathscr{H})$ defined by

$$\mathscr{F}(\mathscr{H}) = \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H})_s \oplus (\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H})_s \oplus ... \tag{16}$$

where the subscript s denotes the symmetric tensor product.

We shall use the following notation: elements of the symmetrized tensor product of n copies of $\mathscr{H}$ will be denoted by Greek letters with n latin upper indices (e.g. $\xi^a \in \mathscr{H}$ and $\xi^{abc} \in (\mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H})_s$, but we can also write $\xi \in \mathscr{H}$ for one particle states if no confusion will arise), while elements of the symmetrized tensor product of n copies of $\bar{\mathscr{H}}$ are denoted by barred Greek letters with n lower latin indices (e.g. $\bar{\xi}_a \in \bar{\mathscr{H}}$, or $\bar{\xi} \in \bar{\mathscr{H}}$). We will write an element $\psi \in \mathscr{F}(\mathscr{H})$ as

$$\psi = (c, \xi^a, \xi^{ab}, \xi^{abc}, ...). \tag{17}$$

A contraction of indices, e.g. $\xi^a \bar{\sigma}_a$, will denote the complex number obtained by applying $\bar{\sigma}_a$ to ξ^a. In the following, we will consider the standard scalar product on $\mathscr{H}$,

$$(\sigma, \xi) = \xi^a \bar{\sigma}_a. \tag{18}$$

For every $\bar{\tau} \in \bar{\mathscr{H}}$ we define the *annihilation operator* $a(\bar{\tau}) : \mathscr{F}(\mathscr{H}) \to \mathscr{F}(\mathscr{H})$ by

$$a(\bar{\tau})\psi = (\xi^a \bar{\tau}_a, \sqrt{2}\xi^{ab}\bar{\tau}_a, \sqrt{3}\xi^{abc}\bar{\tau}_a, ...) \tag{19}$$

where ψ is defined in (17). Similarly, for every $\tau \in \mathscr{H}$, we define the *creation operator* $a^\dagger(\tau)$: $\mathscr{F}(\mathscr{H}) \to \mathscr{F}(\mathscr{H})$ by

$$a^\dagger(\tau)\psi = (0, c\tau^a, \sqrt{2}\tau^{(a}\xi^{b)}, \sqrt{3}\tau^{(a}\xi^{bc)}, ...), \tag{20}$$

where

$$\tau^{(a}\xi^{bc...z)} = \sqrt{\frac{1}{n!}} \sum_{\sigma \in S_n} \tau^{\sigma(a)}\xi^{\sigma(b)\sigma(c)...\sigma(z)}, \tag{21}$$

where the set $\{a, b, c, ..., z\}$ contains n elements and S_n is the group of permutations of the elements of this set.

Before proceeding further, it is useful to discuss some features of the space of solutions of the classical Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)F(x) = 0. \tag{22}$$

On this space, the expression

$$(F, G)_{KG} = i \int_\Sigma (\bar{F}\partial_\mu G - G\partial_\mu \bar{F}) d\Sigma^\mu, \tag{23}$$

where Σ is an asymptotically flat spacelike hypersurface, defines a scalar product. One can show that this scalar product is independent of the choice of Σ. The space of positive frequency solutions of finite Klein-Gordon norm $\mathscr{K}$ is isomorphic to $\mathscr{H}$, and the correspondence is given by

$$\hat{F}(k^\mu) = \sigma_F(k^\mu)\delta(k^\nu k_\nu + m^2) \tag{24}$$

where $\hat{F}$ denotes the fourier transform of F and σ_F indicates the element of $\mathscr{H}$ corresponding to F under this isomorphism. Moreover, one can show that

$$(F, G)_{KG} = (\sigma_F, \sigma_G) \tag{25}$$

where the scalar product on the right is the one defined in (18). Similarly, the space of negative frequency solutions of finite Klein-Gordon norm $\bar{\mathscr{K}}$ is isomorphic to $\bar{\mathscr{H}}$ and one can show that

$$(\bar{F}, \bar{G})_{KG} = -(\bar{\sigma}_F, \bar{\sigma}_G) = -(\sigma_G, \sigma_F). \tag{26}$$

As shown in [2], the previous definitions allow us to express the operator ϕ satisfying equation (15) as

$$\phi = \sum_i (F_i a_i + \bar{F}_i a_i^\dagger), \tag{27}$$

where $\{F_i\}$ is an orthonormal basis of $\mathscr{K}$, $\sigma_i = \sigma_{F_i}$, $\bar{\sigma}_i = \sigma_{\bar{F}_i}$, $a_i = a(\bar{\sigma}_i)$ and $a_i^\dagger = a^\dagger(\sigma_i)$.

3.2 Quantum field theory in an external potential or curved, asymptotically flat spacetime (no horizon)

It is now necessary to develop a quantum field theory in a curved spacetime (where a potential could also be present) which can be later applied to the case of the Schwarzschild solution, in order to study the phenomenon of particle creation by black holes. We will start by considering a situation where no horizon is present and we will develop a theory, similar to a scattering theory, where the metric and the potential will be treated classically while the matter fields will be treated quantum mechanically.

We will start by making some minimal assumptions on the spacetime background and on the field operators. This will allow us to uniquely determine the S-matrix S associated to this scattering theory and the state $\Psi = S\Psi_0$, where Ψ_0 represents the vacuum state in the asymptotic past and Ψ is its image under S living in the asymptotic future. Ψ will therefore contain information on the particle creation process.

3.2.1 Introducing some minimal assumptions on the metric

In this section we will consider the quantum field theory associated with the operator ϕ satisfying the equation

$$(\nabla_\mu \nabla^\mu + m^2 + V(x))\phi(x) = 0, \tag{28}$$

where ∇_μ denotes the covariant derivative and V is some potential. We will assume that the spacetime curvature and V have compact support. Equation (28) is a generalization of the Klein-Gordon equation in curved spacetime and reduces to equation (15) outside the union of the supports of V and the curvature. Moreover, one can show that

$$(F, G)_{KG} = i \int_\Sigma (\bar{F} \nabla_\mu G - G \nabla_\mu \bar{F}) d\Sigma^\mu, \tag{29}$$

defines a scalar product in this space. In expression (29), Σ indicates an asymptotically flat spacelike hypersurface and one can show that the value of $(F, G)_{KG}$ is independent of the choice of Σ. Moreover, outside of the support of V and the curvature, expression (29) reduces to expression (23).

3.2.2 Introducing some minimal assumptions on the field operators

The next step is to introduce some assumptions on the operator ϕ. We will denote with $\mathscr{F}$ the Hilbert space on which ϕ acts. In the far past and in the far future (more precisely, outside the union of supports of V and the curvature), we want the states of $\mathscr{F}$ to look like the states of the free field $\mathscr{F}(\mathscr{H})$, defined in (16). If we denote with $\mathscr{F}_{in}(\mathscr{H})$ and $\mathscr{F}_{out}(\mathscr{H})$ two copies of $\mathscr{F}(\mathscr{H})$, we can state this assumption more precisely by requiring there to be isomorphisms $U : \mathscr{F} \to \mathscr{F}_{in}(\mathscr{H})$ and $W : \mathscr{F} \to \mathscr{F}_{out}(\mathscr{H})$ such that we have

$$U\phi U^{-1} = \phi_{in} = \sum_i (G_i a_i + \bar{G}_i a_i^\dagger) \tag{30}$$

and

$$W\phi W^{-1} = \phi_{out} = \sum_j (H_j b_j + \bar{H}_j b_j^\dagger). \tag{31}$$

Here, a and $a^\dagger$ denote the annihilation and creation operators on $\mathscr{F}_{in}(\mathscr{H})$, while b and $b^\dagger$ denote the annihilation and creation operators on $\mathscr{F}_{out}(\mathscr{H})$. G_i are solutions of the classical version of equation (28) which agree in the past with the free solution F_i appearing in equation (27) (more precisely, G_i may be constructed by choosing a spacelike hypersurface which lies entirely outside the support of the curvature and V in the past and assigning the value and time deivative of F_i on that slice as initial data for a solution of the classical equation (28)). Similarly, H_i are solutions of the classical version of (28) which agree in the far future with F_i. $\mathscr{F}_{in}(\mathscr{H})$ and $\mathscr{F}_{out}(\mathscr{H})$ can be interpreted as the spaces of incoming and outgoing particle states. The S-matrix, defined as $S = WU^{-1}$, relates $\mathscr{F}_{in}(\mathscr{H})$ to $\mathscr{F}_{out}(\mathscr{H})$ and therefore gives

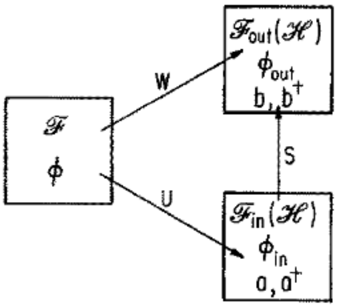

Figure 4: The relationships between the Hilbert space of states $\mathscr{F}$ and the "in" and "out" Hilbert spaces $\mathscr{F}_{in}(\mathscr{H})$ and $\mathscr{F}_{out}(\mathscr{H})$. This picture is taken from [2].

all the relevant information concerning the scattering process. The situation is depicted in figure 4.

By left-multiplying equation (30) by S and right-multiplying it by S^{-1} and by comparing it to equation (31), we obtain

$$S\left[\sum_i \left(G_i a_i + \bar{G}_i a_i^\dagger\right)\right] S^{-1} = \sum_i \left(H_j b_j + \bar{H}_j b_j^\dagger\right), \tag{32}$$

which is equivalent to

$$\sum_i \left[G_i(S a_i S^{-1}) + \bar{G}_i(S a_i^\dagger S^{-1})\right] = \sum_i \left(H_j b_j + \bar{H}_j b_j^\dagger\right). \tag{33}$$

We now take the scalar product of G_n with both sides of (33) and obtain

$$(G_n, \sum_i \left[G_i(S a_i S^{-1}) + \bar{G}_i(S a_i^\dagger S^{-1})\right])_{KG} = (G_n, \sum_i \left(H_j b_j + \bar{H}_j b_j^\dagger\right))_{KG}, \tag{34}$$

which is equivalent to

$$\sum_i \left[(G_n, G_i)_{KG}(S a_i S^{-1}) + (G_n, \bar{G}_i)_{KG}(S a_i^\dagger S^{-1})\right] = \sum_i \left((G_n, H_j)_{KG} b_j + (G_n, \bar{H}_j)_{KG} b_j^\dagger\right). \tag{35}$$

Since $\{G_i\}$ is an orthonormal basis of the space of the solutions of the classical Klein-Gordon equation, and since the space of positive frequency solutions and the space of negative frequency solutions are orthogonal to each other, we have that $(G_n, G_i)_{KG} = \delta_{ni}$ and that $(G_n, \bar{G}_i)_{KG} = 0$. Therefore, we obtain

$$S a_n S^{-1} = \sum_i \left((G_n, H_j)_{KG} b_j + (G_n, \bar{H}_j)_{KG} b_j^\dagger\right). \tag{36}$$

Equation (36) illustrates very well the mixing of creation and annihilation operators due to the scattering.

3.2.3 Defining some scattering operators

We will now introduce some operators which will allow us to rewrite (36) in a cleaner way. These operators will contain information about the mixing of positive and negative frequencies due to the scattering. Since we will refer to [2] for some proofs, we will use the same notation as there.

Let F be a positive frequency solution of the classical free Klein-Gordon equation (22). Let G be a solution of the classical version of equation (28) which agrees with F in the far past. In the future, G will agree with some solution of (22) (which can be a superposition of positive and negative frequency solutions). Decomposing this classical free field solution into its positive and negative components, we can uniquely write

$$G = H' + \bar{H}'', \tag{37}$$

where H' and H'' are solutions of the classical version of equation (28) which agree in the future with positive frequency free field solutions, denoted F' and F'' respectively. We define the operators $C : \mathscr{H} \to \mathscr{H}$ and $D : \mathscr{H} \to \bar{\mathscr{H}}$ by

$$\boxed{C\sigma_F = \sigma_{F'}} \tag{38}$$

and

$$\boxed{D\sigma_F = \bar{\sigma}_{F''}.} \tag{39}$$

The situation is depicted in figure 5. Moreover, we define the operators $\bar{C} : \bar{\mathscr{H}} \to \bar{\mathscr{H}}$ and $\bar{D} : \bar{\mathscr{H}} \to \mathscr{H}$ by

$$\bar{C}\bar{\sigma} = \overline{C\sigma} \tag{40}$$

and

$$\bar{D}\bar{\sigma} = \overline{D\sigma}. \tag{41}$$

We are now in position to rewrite equation (36) in a more compact form. For all states $\sigma \in \mathscr{H}$, we have

$$Sa(\bar{\sigma})S^{-1} = b(\overline{C\sigma}) - b^\dagger(\overline{D\sigma}). \tag{42}$$

A detailed proof for this can be found in section III of [2]. Setting $\tau = C\sigma$ and defining the operator $E : \bar{\mathscr{H}} \to \mathscr{H}$ by

$$\boxed{E = \bar{D}\bar{C}^{-1},} \tag{43}$$

we have for all $\tau \in \mathscr{H}$

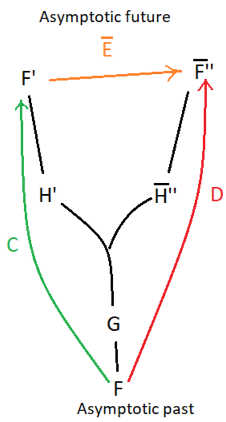

Figure 5: The action of the operators C, D and E.

$$Sa(\overline{C^{-1}\tau})S^{-1} = b(\bar{\tau}) - b^\dagger(E\bar{\tau}). \tag{44}$$

Given a solution of (28) of the form (37), the operator $\bar{E}$ maps the one-particle state associated with the positive frequency part to the one associated with the negative frequency part after the scattering.

Since all the necessary operators have been introduced, we can now proceed to the next step, consisting of determining the image of the incoming vacuum state under the S-matrix, using the result (44).

3.2.4 Determining $\Psi = S\Psi_0$

We begin by defining

$$\boxed{\Psi = S\Psi_0,} \tag{45}$$

where Ψ_0 is the incoming vacuum state ($\Psi_0 \in \mathscr{F}_{in}(\mathscr{H})$). Physically, Ψ contains complete information on particle creation from the vacuum. By right-multiplying both sides of equation (44) by Ψ, we obtain

$$Sa(\overline{C^{-1}\tau})\Psi_0 = \left[b(\bar{\tau}) - b^\dagger(E\bar{\tau})\right]\Psi. \tag{46}$$

Since a is the annihilation operator on $\mathscr{F}_{in}(\mathscr{H})$ and $\Psi_0 \in \mathscr{F}_{in}(\mathscr{H})$ is the vacuum state of $\mathscr{F}_{in}(\mathscr{H})$, we have that $a(\overline{C^{-1}\tau})\Psi_0 = 0$. Equation (46) then becomes

$$b(\bar{\tau})\Psi = b^\dagger(E\bar{\tau})\Psi. \tag{47}$$

Writing

$$\Psi = (c, \eta^a, \eta^{ab}, \eta^{abc}, \eta^{abcd}, ...) \tag{48}$$

and solving equation (47) component by component (using the definition of creation and annihilation operators defined in (19) and (20)), we find for the first four terms

$$\eta^a\bar{\tau}_a = 0 \tag{49a}$$

$$\sqrt{2}\eta^{ab}\bar{\tau}_a = c(E\bar{\tau})^b \tag{49b}$$

$$\sqrt{3}\eta^{abc}\bar{\tau}_a = \sqrt{2}(E\bar{\tau})^{(b}\eta^{c)} \tag{49c}$$

$$\sqrt{4}\eta^{abcd}\bar{\tau}_a = \sqrt{3}(E\bar{\tau})^{(b}\eta^{cd)}. \tag{49d}$$

Since these equations need to be satisfied for all $\tau \in \mathscr{H}$, it follows from (49a) that $\eta^a = 0$ (η^a is the zero vector in the vector space $\mathscr{H}$). Equation (49c) then implies that $\eta^{abc} = 0$. By induction, it follows that $\eta^{\overbrace{abc...z}^{n}} = 0$ for n odd. This means that the amplitude for being in a state with an odd number of particles vanishes, i.e. particles are created in pairs. Equation (49b) states that E and η^{ab}, seen as operators from $\bar{\mathscr{H}}$ to $\mathscr{H}$, must be proportional. This means that there exists[1] a state $\epsilon^{ab} \in (\mathscr{H} \otimes \mathscr{H})_s$ such that

$$\epsilon^{ab}\bar{\xi}_a = (E\bar{\xi})^b \tag{50}$$

for every $\xi \in \mathscr{H}$ and such that

$$\eta^{ab} = (c/\sqrt{2})\epsilon^{ab}. \tag{51}$$

[1] As explained in [2], in order for such a state to exist, the operator E must satisfy two conditions. It must be a symmetric operator and it must satisfy the condition $tr(E^\dagger E) < \infty$ in order for the scattering theory to exist. If these conditions are satisfied, one can show that the norm of $\Psi = S\Psi_0$ is finite. The behavior of E depends on the choice of the field equations and on the spacetime metric. In the case of a Klein-Gordon field on a Schwarzschild background, these conditions are satisfied, as shown in [2]. Therefore, we do not need to care about them.

Equation (49d) then yields

$$\eta^{abcd} = c((3 \cdot 1)/(4 \cdot 2))^{1/2} \epsilon^{(ab} \epsilon^{cd)} \tag{52}$$

and by induction, we obtain for the n-particle state (n even)

$$\eta^{abcd...yz} = c((2n)!^{1/2}/(2^n \cdot n!)) \epsilon^{(ab} \epsilon^{cd} ... \epsilon^{yz)}. \tag{53}$$

Therefore, the state $\Psi = S\Psi_0$ has following form

$$\boxed{\Psi = \Psi(\epsilon^{ab}) = c(1, 0, 2^{-1/2}\epsilon^{ab}, 0, ((3 \cdot 1)/(4 \cdot 2))^{1/2} \epsilon^{(ab}\epsilon^{cd)}, 0, ...).} \tag{54}$$

We can choose c to make $\|\Psi\| = 1$.

In order for the theory to be consistent, it remains to show that the remainder of the S-matrix of this theory is uniquely determined and that the S-matrix is unitary (i.e., that $S^{-1} = S^{\dagger}$). A proof for this can be found in section III of [2].

In summary, a consistent theory satisfying our initial requirements does exist. Using this theory, we found that particles are created in pairs. In the next section, we will apply this theory to the Klein-Gordon field on a Schwarzschild background in order to quantify the particle creation by black holes.

4 Particle creation by gravitational collapse

We have now reached the main point of this work, where we will try to apply the theory developed in the previous section to the Klein-Gordon field together with the Schwarzschild solution. First of all, we will show that the Schwarzschild solution satisfies the minimal requirements for this theory to be applied. Then, we will find what the operator E (and therefore the state ϵ^{ab}) looks like. This will allow us to determine the state $\Psi = S\Psi_0$ which results when gravitational collapse of a body occurs with no particles initially present (i.e. starting with the vacuum incoming state Ψ_0). Finally, we will determine the expectation value of the number of particles $\langle \mathscr{N} \rangle$ observed at late times (i.e. when the black hole has settled down).

4.1 The classical Klein-Gordon field on the Schwarzschild background

We will start by considering the Schwarzschild extended spacetime of figure 2. The classical Klein-Gordon equation looks

$$(\nabla_\mu \nabla^\mu + m^2)\phi = 0, \tag{55}$$

where ∇_μ is the covariant derivative associated to the Schwarzschild metric. Since the Schwarzschild spacetime has spherical symmetry, it makes sense to expand ϕ in spherical harmonics

$$\phi(t, r_*, \theta, \psi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{f_{lm}(r_*, t)}{r} Y_{lm}(\theta, \psi), \tag{56}$$

where $f_{lm}(r_*, t)$ is a function of r_* and t (Regge-Wheeler coordinates). By plugging (56) into (55), we obtain

$$\frac{\partial^2 f_{lm}}{\partial t^2} - \frac{\partial^2 f_{lm}}{\partial r_*^2} + \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right] f_{lm} = 0, \tag{57}$$

which we can also write as

$$\frac{\partial^2 f_{lm}}{\partial t^2} - \frac{\partial^2 f_{lm}}{\partial r_*^2} + V_l(r) f_{lm} = 0, \tag{58}$$

where

$$V_l(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right], \tag{59}$$

with $r = r(r_*)$, has limits

$$V_l(r) \to \begin{cases} 0, & (r_* \to -\infty, \text{ i.e. } r \to 2M) \\ m^2, & (r_* \to +\infty, \text{ i.e. } r \to +\infty). \end{cases} \tag{60}$$

Equation (58) has then the form of the wave equation for a massless scalar field f in a two-dimensional flat spacetime with a scalar potential (59). If we set $m = 0$, the solution f_{lm} of (58) in the asymptotic past has the form

$$f_{lm,-}(t, r_*) = f_-(t - r_*) + g_-(t + r_*) = f_-(u) + g_-(v) \tag{61}$$

where f_- and g_- approach a free solution (i.e. a solution of (58) where $V_l(r) = 0$). f_- and g_- describe the part of the wave incoming from the white hole, respectively incoming from $\mathscr{I}^-$. The coordinates u and v are the Eddington-Finkelstein coordinates. Similarly, in the asymptotic future, the solution has the form

$$f_{lm,+}(t, r_*) = f_+(t - r_*) + g_+(t + r_*) = f_+(u) + g_+(v) \tag{62}$$

where f_+ and g_+ approach again a free solution and describe the part of the wave outgoing to $\mathscr{I}^+$, respectively outgoing to the black hole. Therefore, in Regge-Wheeler coordinates, the situation looks like the one described in the previous section, where no curvature but a potential is present, and the solutions of the Klein-Gordon equation approach free solutions in the asymptitic past and future. Therefore, the theory developed in the previous section can be applied to the massless Klein-Gordon field in these coordinates. Moreover, a massless Klein-Gordon field in Schwarzschild spacetime is determined by its boundary conditions (i.e. by specifying the free solutions which it approaches), or "data", on the white hole horizon and $\mathscr{I}^-$ or, equivalently, by its data on the black hole horizon and $\mathscr{I}^+$. In the massive case, i.e. $m \neq 0$, these nice results do not occur in Regge-Wheeler coordinates, because at $r \to +\infty$ the solution of (58) does not look like a free solution, since it is distorted by a non vanishing potential. In order to take advantage of the nice properties that the massless Klein-Gordon field has in Regge-Wheeler coordinates, we will explicitly treat the massless Klein-Gordon case below. The Klein-Gordon equation then becomes

$$\nabla_\mu \nabla^\mu \phi = 0. \tag{63}$$

However, all the results should apply to the massive case, as well as to other fields propagating in the Schwarzschild background (see [13] for a detailed discussion).

4.2 Defining the Hilbert spaces of incoming and outgoing one-particle states

So far, we have have made sure that, if we work in Regge-Wheeler coordinates, we can directly apply the theory developed in the previous section to the massless Klein-Gordon field. The next step is to define the Hilbert space of incoming and outgoing one-particle states. Since we have seen that solutions of the massless Klein-Gordon equation can be put into correspondence with functions on the white hole horizon together with functions on $\mathscr{I}^-$, and because of the correspondence (24), we expect the one-particle Hilbert space $\mathscr{H}_{in}$ of incoming states to be the direct sum of a Hilbert space of particles incoming from infinity and a Hilbert space of particles incoming from the white hole

$$\mathscr{H}_{in} = \mathscr{H}_{in,\infty} \oplus \mathscr{H}_{in,wh}. \tag{64}$$

At $\mathscr{I}^-$, there exists a time-translation Killing vector field. Hence, the time translation parameter is well defined and we have a well defined notion of positive frequency solutions, namely those solutions whose Fourier transform on $\mathscr{I}^-$ with respect to t (or v) contain only positive frequencies. This allows us to define the space $\mathscr{H}_{in,\infty}$ unambiguously. However, it is not possible to unambiguously define the notion of positive frequency solution on the white hole, because on the white hole horizon the Killing vector field is light-like. This means that there does not exist a well defined time translation parameter with respect to which one can take the Fourier transform. The Fourier transform could be taken for example with respect to the Schwarzschild retarded time coordinate u or the Kruskal retarded time U (defined in (11)). Since these parameters are related by $U = -\exp(-\kappa u)$, this leads to distinct notions of positive frequency, because in the two cases the space of positive frequency solutions do not coincide. For the same reason, there is ambiguity in the definition of the Hilbert space $\mathscr{H}_{out,bh}$ of particles propagating into the black hole, and hence in the definition of

$$\mathscr{H}_{out} = \mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}. \tag{65}$$

However, as we will see next, it is possible to make physical predictions which avoid this ambiguity.

First, ambiguities in the definition of $\mathscr{H}_{in}$ can be eliminated by replacing the extended Schwarzschild spacetime by the spacetime appropriate to a collapsing spherical body (Figure 3), which describes black holes occurring in nature. This eliminates the white hole horizon and makes $\mathscr{H}_{in}$ be simply $\mathscr{H}_{in,\infty}$, which is unambiguously defined.

Ambiguities in the definition of $\mathscr{H}_{out}$ do not play an important role in our case, since we are interested in the results of measurements made at infinity. To see this, we need the following result.

Theorem. *The spaces $\mathscr{F}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ and $\mathscr{F}(\mathscr{H}_1) \otimes \mathscr{F}(\mathscr{H}_2)$ are isomorphic and for $\Psi_1 \otimes \Psi_2$, $\Psi_3 \otimes \Psi_4 \in \mathscr{F}(\mathscr{H}_1) \otimes \mathscr{F}(\mathscr{H}_2)$ the following expression defines a scalar product on $\mathscr{F}(\mathscr{H}_1) \otimes \mathscr{F}(\mathscr{H}_2)$:*

$$(\Psi_1 \otimes \Psi_2, \Psi_3 \otimes \Psi_4)_{\mathscr{F}(\mathscr{H}_1)\otimes\mathscr{F}(\mathscr{H}_2)} = (\Psi_1, \Psi_3)_{\mathscr{F}(\mathscr{H}_1)} \cdot (\Psi_2, \Psi_4)_{\mathscr{F}(\mathscr{H}_2)},$$

where $(\cdot,\cdot)_{\mathscr{F}(\mathscr{H}_1)\otimes\mathscr{F}(\mathscr{H}_2)}$ is the standard scalar product on $\mathscr{F}(\mathscr{H}_1)\otimes\mathscr{F}(\mathscr{H}_2)$ and $(\cdot,\cdot)_{\mathscr{F}(\mathscr{H}_i)}$ denotes the standard scalar product on $\mathscr{F}(\mathscr{H}_i)$.

Therefore, one can view a state $\Psi \in \mathscr{F}(\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh})$ as a state $\Psi_1 \otimes \Psi_2 \in \mathscr{F}(\mathscr{H}_{out,\infty}) \otimes \mathscr{F}(\mathscr{H}_{out,bh})$ where Ψ_1 contains all information about particles propagating at infinity and Ψ_2 contains information about "particles" propagating into the black hole (where the notion of particle depends on the arbitrary choice of positive frequency solutions on the black hole horizon). Any measurement at infinity can be represented by an operator O on $\mathscr{F}(\mathscr{H}_{out,\infty}) \otimes \mathscr{F}(\mathscr{H}_{out,bh})$

of the form $O = \tilde{O} \otimes I_1$, where I_1 is the identity operator on $\mathscr{F}(\mathscr{H}_{out,bh})$ and $\tilde{O}$ is a Hermitian operator on $\mathscr{F}(\mathscr{H}_{out,\infty})$. Therefore, we have that

$$(\Psi, O\Psi)_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})} \tag{66}$$
$$=(\Psi_1 \otimes \Psi_2, (\tilde{O}\Psi_1) \otimes \Psi_2)_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})} \tag{67}$$
$$=(\Psi_1, \tilde{O}\Psi_1)_{\mathscr{F}(\mathscr{H}_{out,\infty})} \cdot (\Psi_2, \Psi_2)_{\mathscr{F}(\mathscr{H}_{out,bh})} \tag{68}$$
$$=(\Psi_1, \tilde{O}\Psi_1)_{\mathscr{F}(\mathscr{H}_{out,\infty})}. \tag{69}$$

The last step follows from the fact that we normalize states $\Psi_2 \in \mathscr{F}(\mathscr{H}_{out,bh})$ such that $\|\Psi_2\| = 1$. A change in the definition of positive frequency on the black hole horizon will induce a trasformation on the creation and annihilation operators associated with the states representing particles which enter the black hole of the form (42), but will leave unchanged the creation and annihilation operators associated with the states representing particles which propagate to infinity. This will cause the expression for the state Ψ to change to $\Psi' = S\Psi$, where S has the form $S = I_2 \otimes \tilde{S}$, where I_2 is the identity on $\mathscr{F}(\mathscr{H}_{out,\infty})$ and $\tilde{S}$ is a unitary operator on $\mathscr{F}(\mathscr{H}_{out,bh})$. It follows then that

$$(\Psi', O\Psi')_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})} \tag{70}$$
$$=(S\Psi, OS\Psi)_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})} \tag{71}$$
$$=(\Psi_1 \otimes (\tilde{S}\Psi_2), (\tilde{O}\Psi_1) \otimes (\tilde{S}\Psi_2))_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})} \tag{72}$$
$$=(\Psi_1, \tilde{O}\Psi_1)_{\mathscr{F}(\mathscr{H}_{out,\infty})} \cdot ((\tilde{S}\Psi_2), (\tilde{S}\Psi_2))_{\mathscr{F}(\mathscr{H}_{out,bh})} \tag{73}$$
$$=(\Psi_1, \tilde{O}\Psi_1)_{\mathscr{F}(\mathscr{H}_{out,\infty})} \cdot (\Psi_2, \Psi_2)_{\mathscr{F}(\mathscr{H}_{out,bh})} \tag{74}$$
$$=(\Psi_1, \tilde{O}\Psi_1)_{\mathscr{F}(\mathscr{H}_{out,\infty})} \tag{75}$$
$$=(\Psi, O\Psi)_{\mathscr{F}(\mathscr{H}_{out,\infty})\otimes\mathscr{F}(\mathscr{H}_{out,bh})}. \tag{76}$$

The fourth step follows from the fact that $\tilde{S}$ is unitary, while the last step follows from the previous calculation. Thus, predictions of the theory with regard to measurements made at infinity must be independent of the definition of positive frequency on the horizon. As a consequence, if we are concerned only with the results of measurements made at infinity, we can arbitrarily define the set of positive frequency solutions on the black hole horizon in such a way that it is convenient for our calculations.

4.3 Postulating the field operators

We are now in position to postulate the field operators acting on $\mathscr{F}(\mathscr{H}_{in})$ and $\mathscr{F}(\mathscr{H}_{out})$. Since at $\mathscr{I}^-$, at $\mathscr{I}^+$ and on the black hole horizon the solutions of the massless classical Klein-Gordon equation asymptotically approach free solutions, the states of the system should asymptotically look like states of the free field Hilbert space $\mathscr{F}_{in}(\mathscr{H})$ and $\mathscr{F}_{out}(\mathscr{H})$. As in the previous section, these assuptions allow to postulate the field operator

$$U\phi U^{-1} = \sum_i (G_i a_i + \bar{G}_i a_i^\dagger) \tag{77}$$

where G_i is the solution of equation (63) with the same data at $\mathscr{I}^-$ as the free field solution F_i, $a_i = a_{F_i}$ and $U : \mathscr{F}(\mathscr{H}_{in}) \to \mathscr{F}_{in}(\mathscr{H})$ is an isomorphism. Similarly, we postulate

$$W\phi W^{-1} = \sum_i (H_i b_i + \bar{H}_i b_i^\dagger + K_i c_i + \bar{K}_i c_i^\dagger) \tag{78}$$

where H_i is the solution of (63) with the same data at $\mathscr{I}^+$ as F_i and vanishing data on the horizon, where $\{K_i\}$ is a set of solutions of (63) which vanish on $\mathscr{I}^+$ (we call it the set of "positive frequency solutions at the black hole horizon"), such that $\{K_i\}$ and their complex conjugates $\{\bar{K}_i\}$ span all solutions which vanish on $\mathscr{I}^+$. Here, c_i and $c_i^\dagger$ are the annihilation and creation operators on $\mathscr{F}(\mathscr{H}_{out,bh})$ associated with the "positive frequency" solution K_i and $W : \mathscr{F}(\mathscr{H}_{out}) \to \mathscr{F}_{out}(\mathscr{H})$ is an isomorphism.

4.4 Determining $\Psi = S\Psi_0$

The mathematical structure of the theory described so far is the same as in the previous section, and we can directly use the results of the analysis given there for the outgoing state $\Psi \in \mathscr{F}(\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh})$ (equation (54)), where ϵ^{ab} is the two-particle state associated with the operator $E = \overline{DC^{-1}}$, where the operators $C : \mathscr{H}_{in} \to \mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$ and $D : \mathscr{H}_{in} \to \overline{\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}}$ are defined as before, except for the fact that now "in the asymptotic future" means "at $\mathscr{I}^+$ and on the back hole" and "in the asymptotic past" now means "at $\mathscr{I}^-$". Our purpose is now to determine the E, and consequently the two-particle state ϵ^{ab}. This is done by determining how the operator E acts on all elements of a basis of $\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$. The process is described step by step below.

1. **Introducing an orthonormal basis of $\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$**
   For every ω, l, m we denote with $P_{\omega lm}$ the free classical solution generated by the data $\omega^{-1/2}\exp(i\omega u)Y_{lm}(\theta, \psi)$ at $\mathscr{I}^+$ associated with the positive frequency ω. In the following, due to the definitions of the operators C and D (equations (38) and (39)), we will have to deal with one-particle states associated with this classical solution. We could think of using $P_{\omega lm}$ and correspondence (24), but since we will have to deal with states propagating in the spacetime, it could be useful to approximate such correspondence by building wavepackets with a very small frequency spread, and therefore associate the one-particle state with these wavepackets. Let us fix a real number L with $0 < L \ll 1$ and define
$$P_{jnlm} = L^{-1/2} \int_{jL}^{(j+1)L} \exp(-2\pi i n\omega'/L) P_{\omega' lm} d\omega'. \tag{79}$$
   then the set $\{P_{jnlm}\}$ with $j \geq 0$ corresponds to an orthonormal basis of $\mathscr{H}_{out,\infty}$ with respect to the Klein-Gordon scalar product. These wavepackets are made up of frequencies within L of $\omega = jL$. They are peaked around the retarded time $u = 2\pi n/L$ and have a time spread $\sim 2\pi/L$. We will use the symbol ${}_i\rho^a$ to denote the element of $\mathscr{H}_{out,\infty}$ corresponding to the wavepacket P_{jnlm}, where the index i stands for $jnlm$.

   We can construct a basis $\{Q_{jnlm}\}$ of $\mathscr{H}_{out,bh}$ using the same procedure, starting from the free "positive frequency" solutions $Q_{\omega lm}$ generated by the data $\omega^{-1/2}\exp(i\omega v)Y_{lm}$ on the black hole horizon. We use the symbol ${}_i\sigma^a$ to denote the basis element corresponding to Q_{jnlm}. The union of the two sets $\{{}_i\rho^a\}$ and $\{{}_i\sigma^a\}$ gives then an orthonormal basis of $\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$.

2. **Constructing wave packets with postive frequency data at $\mathscr{I}^-$**
   If we observe definitions (38), (39) and (43), we notice that, in order to determine the action of operator E on elements of $\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$, we need to work with wave packets generated by purely positive frequency data $L^{-1/2} \int_{jL}^{(j+1)L} \exp(-2\pi i n\omega'/L)\omega'^{-1/2}\exp(i\omega' v)Y_{lm} d\omega'$ at $\mathscr{I}^-$. Therefore, we proceed as follows.

- **Wave packets with data at late advanced times at $\mathscr{I}^-$**
We consider the solutions P_{jnlm} and Q_{jnlm} at late retarded and advanced times (i.e. for large n) and we prescribe data at late advanced times at $\mathscr{I}^-$ for the solution

$$Y_{jnlm} = RP_{jnlm} + TQ_{jnlm}, \tag{80}$$

where $T = T_{lm}(\omega) =: T_i$ and $R = R_{lm}(\omega) =: R_i$ denote the trasmission and reflection amplitudes, i.e. the amplitude for the solution Y_{jnlm} to be trasmitted into the black hole and to be reflected to $\mathscr{I}^+$. We denote the corresponding elements of $\mathscr{H}_{out,\infty} \oplus \mathscr{H}_{out,bh}$ as

$$_i\gamma^a = T_i\, {}_i\sigma^a + R_i\, {}_i\rho^a. \tag{81}$$

Since the operator $\bar{E}$ maps the part of the $_i\gamma^a$ associated to positive frequency data at $\mathscr{I}^+$ and on the black hole horizon to the part of $_i\gamma^a$ associated to negative frequency data at $\mathscr{I}^+$ and on the black hole horizon, it is clear that

$$\boxed{DC^{-1}\, {}_i\gamma^a = \bar{E}\, {}_i\gamma^a = 0.} \tag{82}$$

This is due to the fact that the solution $_i\gamma^a$ is only associated to positive frequency data at $\mathscr{I}^+$ and on the black hole [2], and therefore the part of $_i\gamma^a$ associated to positive frequency data is represented by $_i\gamma^a$ itself, while the part of $_i\gamma^a$ associated to negative frequency data is the zero vector of $\mathscr{H}_{out}$ (the vector which corresponds to the trivial solution of the classical Klein-Gordon equation). It remains now to construct new wave packets associated to data at earlier advanced times on $\mathscr{I}^-$.

- **Wave packets with data at advanced times around $v = 0$ at $\mathscr{I}^-$**
For a moment, we go back to the extended Schwarzschild spacetime. In a similar way as above, we consider the solutions P_{jnlm} and Q_{jnlm} at late retarded and advanced times and we prescribe data at late retarded times on the white hole horizon for the solution

$$X_{jnlm} = tP_{jnlm} + rQ_{jnlm} \tag{83}$$

where $t = t_{lm}(\omega) =: t_i$ and $r = r_{lm}(\omega) =: r_i$ denote the amplitude for the solution X_{jnlm} to be trasmitted to $\mathscr{I}^+$ and to be reflected into the black hole, respectively. We denote the corresponding basis elements as

$$_i\lambda^a = t_i\, {}_i\rho^a + r_i\, {}_i\sigma^a. \tag{84}$$

Keeping this definition in mind, we now move back to the spacetime associated to a collapsing body, where the white hole horizon is replaced by a null geodesic of constant advanced time $v = v_0$. For convenience, we set $v_0 = 0$.

We first consider a free solution ϕ of the classical Klein-Gordon equation associated with the data $\phi_0 \exp(-i\omega u)$ at $\mathscr{I}^+$. Suppose we are interested in how an observer with affine parameter λ on the surface of the collapsing body sees this solution. In order to understand this, let us apply the following coordinate transformation

$$u = -\frac{1}{\kappa}\ln(-U), \tag{85}$$

[2] This happens because the wave packet Y_{jnlm} associated to the state $_i\gamma^a$ gets scattered at late times. For this reason, it propagates through a region of space which is almost flat (and not through the collapsing body). Therefore, the mixing of positive and negative frequencies does not occur.

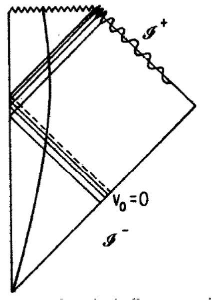

Figure 6: Penrose diagram of gravitational collapse. This picture is taken from [2].

which follows immediately from (11). Let us now set $\lambda = 0$ at the point where the oberver's geodesic crosses the black hole horizon. Since, as one can show, the parameter λ depends smoothly on U and $\frac{dU}{d\lambda}\Big|_{\lambda=0} \neq 0$, and since $U(\lambda = 0) = 0$ [3], one can make a Taylor expansion in the vicinity of $\lambda = 0$ and write

$$U(\lambda) \cong U(\lambda = 0) + \frac{dU}{d\lambda}\Big|_{\lambda=0} \cdot \lambda = \frac{dU}{d\lambda}\Big|_{\lambda=0} \cdot \lambda = \alpha\lambda \tag{86}$$

where $\alpha = \frac{dU}{d\lambda}\Big|_{\lambda=0}$. We can now insert equation (86) into equation (85) and then into the expression $\phi = \phi_0 \exp(-i\omega u)$ and obtain

$$\phi(\lambda) = \phi_0 \cdot \exp\left[\frac{i\omega}{\kappa}\ln(-\alpha\lambda)\right]. \tag{87}$$

Therefore, to our observer, the frequency of the solution appears to diverge as it approaches $\lambda = 0$. This implies that, if we now propagate the solution $\phi_0 \exp(-i\omega u)$ backwards in time, we are allowed to use the ray approximation[4] through the collapsing body in the vicinity of the black hole horizon. Therefore, the solution behaves inside the collapsing body as it would do outside, i.e. it will have the form $\phi_0 e^{iS}$, where the surfaces of constant phase S are null and are represented in the Penrose diagram as straight lines at $\pm 45°$. Hence, the pattern made by the wave near $v = 0$ at $\mathscr{I}^-$ in the diagram can be obtained by continuing the null geodesic generators of the surfaces of constant S back to $\mathscr{I}^-$.

We are now interested in determining the form of the solution ϕ at $\mathscr{I}^-$. To do this, we consider the situation of Figure 7. Let x be a point on the black hole horizon and let n^a be a future-directed null vector a x. Let us denote with λ the affine parameter along a null geodesic parallel to n^a [5]. For small values of λ ($\lambda < 0$), the vector λn^a connects the point x with a nearby null surface of constant retarded time u and therefore with a surface of constant phase of the solution ϕ. Because of the ray approximation, if the vector n^a is parallely transported back in time along the black hole horizon and along the null geodesic of constant advanced time $v = 0$, the vector

[3] This can be seen by noticing that $\lim_{r_* \to -\infty} u(t, r_*) = \lim_{r_* \to -\infty}(t - r_*) = \infty$, i.e. u goes to infinity as one approaches the horizon, and by using (11) to compute $\lim_{u\to\infty} U(u) = \lim_{u\to\infty} -\exp(-\kappa u) = 0$, i.e. U goes to 0 as one approaches the horizon, where $\lambda = 0$.

[4] The ray approximation states that, for sufficiently high frequencies, the rays are straight lines when passing through a medium.

[5] Let us emphasize here that relation (87) is also valid for this kind of geodesic.

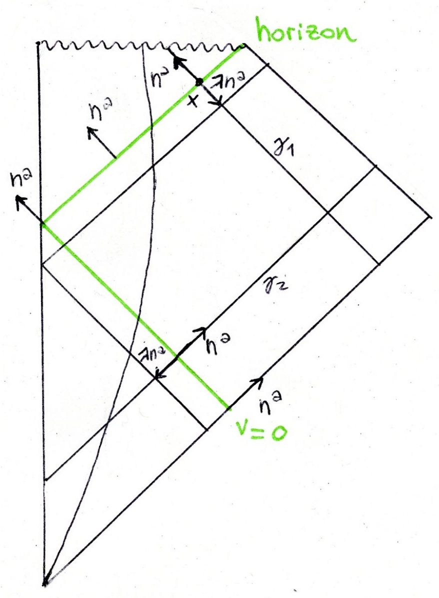

Figure 7: This diagram illustrates the behaviour of the solution ϕ as a function of λ. γ_1 and γ_2 are null geodesics.

λn^a will always connect the event horizon and the geodesic of constant advanced time $v = 0$ to the same surface of constant phase. Therefore, the behaviour of the solution ϕ as a function of λ does not change when n^a is parallely transported. At $\mathscr{I}^-$, n^a will be parallel to the null geodesic generator of $\mathscr{I}^-$. Since $\lambda = 0$ along the geodesic of constant advanced time $v = 0$, we have at $\mathscr{I}^-$ that

$$\lambda(v) = \lambda(v = 0) + \left.\frac{d\lambda}{dv}\right|_{v=0} \cdot v = \left.\frac{d\lambda}{dv}\right|_{v=0} \cdot v = \beta \cdot v \tag{88}$$

where $\beta = \left.\frac{d\lambda}{dv}\right|_{v=0}$. The behaviour of the solution ϕ as a function of v at $\mathscr{I}^-$ is then determined by inserting expression (88) into expression (87). One then obtains, for $v > 0$,

$$\phi(v) = \phi_0 \cdot \exp\left[\frac{i\omega}{\kappa}\ln(-\alpha\beta v)\right] \tag{89}$$

$$= \phi_0 \cdot \exp\left[\frac{i\omega}{\kappa}\left(\ln(-v) + \ln(\alpha\beta)\right)\right] \tag{90}$$

$$= \phi_0 \cdot \exp\left[\frac{i\omega}{\kappa}\ln(-v)\right] \cdot \exp\left[\frac{i\omega}{\kappa}\ln(\alpha\beta)\right] \tag{91}$$

$$= \tilde{\phi_0} \cdot \exp\left[\frac{i\omega}{\kappa}\ln(-v)\right] \tag{92}$$

where $\tilde{\phi_0} = \phi_0 \cdot \exp\left[\frac{i\omega}{\kappa}\ln(\alpha\beta)\right]$. The final result is then

$$\phi(v) = \begin{cases} 0 & v > 0 \\ \tilde{\phi_0} \cdot \exp\left[\frac{i\omega}{\kappa}\ln(-v)\right] & v < 0. \end{cases} \tag{93}$$

We now consider propagating the wave packet associated to the state ${}_i\lambda^a$ backward in time through the collapsing body and back to $\mathscr{I}^-$. One obtains the corresponding wave packet at $\mathscr{I}^-$ by building wave packets of the solution (93). By denoting with Z_{jnlm} this wave packet at $\mathscr{I}^-$, we obtain the following result

$$Z_{jlnm}(v) \sim \begin{cases} 0, & v > 0 \\ \exp(-i\omega J/L)\sin(J/2)/J & v < 0, \end{cases} \tag{94}$$

where $\omega = (j + \frac{1}{2})L$ is the effective frequency of the original wave packet at $\mathscr{I}^+$ and the future horizon and where

$$J = 2\pi n + (L/\kappa)\ln(-v). \tag{95}$$

If we compute the Fourier transform $\hat{Z}_{jnlm}(\Omega)$ of $Z_{jnlm}(v)$ (see [2], Appendix A), we see that for $\Omega > 0$ it satisfies following relation

$$\hat{Z}_{jnlm}(-\Omega) = -\exp(-\pi\omega/\kappa)\hat{Z}_{jnlm}(\Omega). \tag{96}$$

This implies that, although we started with a pure positive frequency wave packet

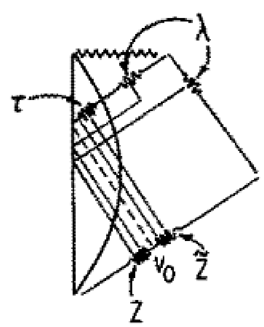

Figure 8: The relationship between ${}_i\lambda^a$, ${}_i\tau^a$, Z_{jnlm} and $\tilde{Z}_{jnlm}$. This picture is taken from [2].

at $\mathscr{I}^+$ and at the black hole horizon, the wave packet Z_{jnlm} at $\mathscr{I}^-$ contains also negative frequency modes. As already seen in equation (36), the mixing of positive and negative frequencies indicates particle production.

Since the wave packet Z_{jnlm} is not of positive frequency at $\mathscr{I}^-$, we need some manipulations before being able to determine the action of the operator E on the one-particle states associated to it at $\mathscr{I}^+$ and on the black hole horizon. First, define the "time inverted" wave packet $\tilde{Z}_{jnlm}$ at $\mathscr{I}^-$ given by

$$\tilde{Z}_{jnlm}(v) = Z_{jnlm}(-v) \sim \begin{cases} \exp(-i\omega\tilde{J}/L)\sin(\tilde{J}/2)/\tilde{J} & v > 0 \\ 0 & v < 0, \end{cases} \tag{97}$$

where

$$\tilde{J} = 2\pi n + (L/\kappa)\ln(v). \tag{98}$$

Since the time inversion changes the sign of the Klein-Gordon scalar product, the $\{\tilde{Z}_{jnlm}\}$ are orthonormal but with negative norm. Moreover, the scalar product of $\tilde{Z}_{jnlm}$ with Z_{jnlm} vanishes. Furthermore, because of time inversion, the Fourier transform $\hat{\tilde{Z}}_{jnlm}(\Omega)$ satisfies

$$\hat{\tilde{Z}}_{jnlm}(-\Omega) = \hat{Z}_{jnlm}(\Omega). \tag{99}$$

Suppose now we propagate the wave packet $\tilde{Z}_{jnlm}$ from $\mathscr{I}^-$ into the future. It would reach the center of the collapsing body just after the formation of the black hole horizon and therefore it would propagate entirely into the black hole. Let J_{jnlm} denote the data for this wave packet at the black hole horizon. We will use our freedom in defining positive frequency at the horizon to take the $\bar{J}_{jnlm}$ as part of our positive frequency basis $\{K_i\}$. Let us denote by ${}_i\tau^a$ the element of $\mathscr{H}_{out}$ associated with the wave packet $\bar{J}_{jnlm}$.

One further step is required before determining the action of the operator E. If we insert equation (99) into equation (96), we obtain

$$\hat{Z}_{jnlm}(-\Omega) + \exp(-\pi\omega/\kappa)\hat{\tilde{Z}}_{jnlm}(-\Omega) = 0, \tag{100}$$

which can be rewritten as

$$(Z_{jnlm}(-\Omega) + \widehat{\exp(-\pi\omega/\kappa)}\tilde{Z}_{jnlm}(-\Omega)) = 0 \tag{101}$$

because of linearity of the Fourier transform. Equation (101) implies that the wavepacket $\hat{Z}_{jnlm}(-\Omega)+\exp(-\pi\omega/\kappa)\hat{\tilde{Z}}_{jnlm}(-\Omega)$ does not contain negative frequency modes. If we now propagate the wave packet associated with the state $({}_i\lambda^a + \exp(-\pi\omega_i/\kappa)\,{}_i\bar{\tau}_a)$ backward into the past, we obtain precisely $\hat{Z}_{jnlm}(-\Omega)+\exp(-\pi\omega/\kappa)\cdot \hat{\tilde{Z}}_{jnlm}(-\Omega)$, which is a purely positive frequency wave packet at $\mathscr{I}^-$. This is precisely the situation which allow us to determine the action of the operator E. Since the state ${}_i\lambda^a$ is associated to a purely positive frequency wave packet, while the state ${}_i\bar{\tau}^a$ is associated to a purely negative frequency wave packet, we deduce that

$$\boxed{DC^{-1}\,{}_i\lambda^a = \bar{E}\,{}_i\lambda^a = \exp(-\pi\omega_i/\kappa)\,{}_i\bar{\tau}_a.} \tag{102}$$

Similarly, if we propagate the wave packet corresponding to the state $({}_i\bar{\lambda}_a + \exp(+\pi\omega_i/\kappa)\,{}_i\tau^a)$ backward into the past, we also obtain a purely positive frequency wave packet at $\mathscr{I}^-$. This implies

$$\boxed{DC^{-1}(\exp(+\pi\omega_i/\kappa)\,{}_i\tau^a) = \bar{E}\exp(+\pi\omega_i/\kappa)\,{}_i\tau^a) =_i \bar{\lambda}_a.} \tag{103}$$

It remains to consider wave packets associated to data at early advanced times on $\mathscr{I}^-$.

- **Wave packets with data at early advanced times on $\mathscr{I}^-$**
  Since we are interested in describing the black hole radiation at late times, i.e. when the black hole has settled down, we are not particularly interested in the form of these states. We simply denote with ϵ_0^{ab} the two particle state associated to the operator E which acts on these states and which is orthogonal to all late times basis vectors (i.e. $\epsilon_0^{ab}\,{}_i\bar{\lambda}_a = 0$, $\epsilon_0^{ab}\,{}_i\bar{\gamma}_a = 0$ and $\epsilon_0^{ab}\,{}_i\bar{\tau}_a = 0$).

3. **Determine the two-particle state ϵ^{ab}**
   We have now determined how the operator $\bar{E}$ acts on the basis of $\mathscr{H}_{out}$ composed by the states ${}_i\gamma^a$, ${}_i\bar{\lambda}_a$, $({}_i\lambda^a + \exp(-\pi\omega_i/\kappa)\,{}_i\bar{\tau}_a)$ as well as states which reach $\mathscr{I}^+$ at early times (see equations (82), (102) and (103)). Therefore, we are now ready to determine the

operator E and the two-particle state ϵ^{ab} associated to it. One can easily see that the state ϵ^{ab} must be of the form

$$\boxed{\epsilon^{ab} = \sum_i \exp(-\pi\omega_i/\kappa)2\, {}_i\lambda^{(a}\, {}_i\tau^{b)} + \epsilon_0^{ab}.} \tag{104}$$

Indeed, we have that

$$\epsilon^{ab}\, {}_i\bar{\gamma}_a = 0, \tag{105}$$
$$\epsilon^{ab}\, {}_i\bar{\lambda}_a = \exp(-\pi\omega_i/\kappa)\, {}_i\tau^a, \tag{106}$$
$$\epsilon^{ab}(\exp(\pi\omega_i/\kappa)\, {}_i\bar{\tau}_a) =_i \lambda^a, \tag{107}$$

because the states ${}_i\lambda^a$, ${}_i\bar{\lambda}_a$, ${}_i\gamma^a$, ${}_i\bar{\gamma}_a$, ${}_i\tau^a$ and ${}_i\bar{\tau}_a$ are orthogonal to each other and to ϵ_0^{ab} are normalized to 1.

Equation (54), together with equation (104), gives the solution for the state vector Ψ which results from particle creation starting from the vacuum during gravitational collapse. The task that remains is to interpret our solution and derive its properties.

4.5 Interpreting the state vector Ψ

We first need to reduce the state vector $\Psi(\epsilon^{ab})$ to a form where it can be easily interpreted. To do this, we use the following lemma.

Lemma. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $\psi^{ab} \in (\mathscr{H}_1 \otimes \mathscr{H}_1)_s$, $\eta^{ab} \in (\mathscr{H}_2 \otimes \mathscr{H}_2)_s$. Consider the state $\phi(\mu^{ab}) \in \mathscr{F}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ defined by*

$$\phi(\mu^{ab}) = (1, 0, 2^{-1/2}\mu^{ab}, 0, ((3\cdot 1)/(4\cdot 2))^{1/2}\mu^{(ab}\mu^{cd)}, 0, ...),$$

where $\mu^{ab} = \psi^{ab} + \eta^{ab}$. Then under the natural isomorphism discussed above between $\mathscr{F}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ and $\mathscr{F}(\mathscr{H}_1) \otimes \mathscr{F}(\mathscr{H}_2)$ the state $\psi(\mu^{ab})$ is mapped into the simple product state $\phi_1(\psi^{ab}) \otimes \phi_2(\eta^{ab})$ where

$$\phi_1(\psi^{ab}) = (1, 0, 2^{-1/2}\psi^{ab}, 0, ((3\cdot 1)/(4\cdot 2))^{1/2}\psi^{(ab}\psi^{cd)}, 0, ...),$$
$$\phi_2(\eta^{ab}) = (1, 0, 2^{-1/2}\eta^{ab}, 0, ((3\cdot 1)/(4\cdot 2))^{1/2}\eta^{(ab}\eta^{cd)}, 0, ...).$$

Proof. See [2], Appendix B. □

We denote with $\mathscr{H}_1 = \operatorname{span}(\bigcup_i \{{}_i\lambda^a,\ {}_i\tau^a\})$, $\mathscr{H}_2 = \operatorname{span}(\bigcup_k \{\, {}_k\gamma^a\})$ and we call $\mathscr{H}_3$ the space spanned by one-particle states corresponding to early times emission. We notice that $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ are orthogonal to each other, and therefore we can write $\mathscr{H}_{out} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3$. If we denote by $\mathscr{H}_{1i} = \operatorname{span}(\{{}_i\lambda^a,\ {}_i\tau^a\})$, $\mathscr{H}_{2k} = \operatorname{span}(\{\, {}_k\gamma^a\})$, we notice that the spaces $\mathscr{H}_{1,i}$ are othogonal for different indices i and that the spaces $\mathscr{H}_{2,k}$ are orthogonal for different indices k. Therefore, we can rewrite $\mathscr{H}_{out}$ as $\mathscr{H}_{out} = (\bigoplus_i \mathscr{H}_{1,i}) \oplus (\bigoplus_k \mathscr{H}_{2,k}) \oplus \mathscr{H}_3$. By applying more times the lemma on the state $\Psi(\epsilon^{ab}) \in \mathscr{F}(\mathscr{H}_{out})$ we obtain

$$\boxed{\Psi(\epsilon^{ab}) = \left(\bigotimes_i \Psi_i(\exp(-\pi\omega_i/\kappa)2_i\lambda^{(a}\, {}_i\tau^{b)})\right) \otimes \left(\bigotimes_k \Psi_k(0)\right) \otimes \Psi(\epsilon_0^{ab})} \tag{108}$$

where $\Psi_l(\eta^{ab}) = (1, 0, 2^{-1/2}\eta^{ab}, 0, ((3\cdot 1)/(4\cdot 2))^{1/2}\eta^{(ab}\eta^{cd)}, 0, ...)$. Let us briefly comment on the three parts of expression (108).

- Each state $\Psi_i(\exp(-\pi\omega_i/\kappa)2_i\lambda^{(a}{}_i\tau^{b)})$ describes multiple pair creation in the mode i at late times (i.e. due to propagation of the solution of the classical Klein-Gordon equation through the collapsing body), in which one particle (${}_i\tau^a$) of each pair enters the black hole while the other particle (${}_i\lambda^a = t_i\, {}_i\rho^a + r_i\, i^a_\sigma$) reaches infinity (with amplitude t_i) or gets scattered back into the black hole (with amplitue r_i).

- Each state $\Psi_k(0)$ represents the vacuum state of $\mathscr{F}(\mathscr{H}_{2,k})$. Indeed, since the wave packet associated to the state ${}_i\gamma^a$ does not pick up any negative frequencies when propagating from $\mathscr{I}^-$ to $\mathscr{I}^+$, the image of the vacuum state under the S-matrix is again the vacuum state.

- The state $\Psi(\epsilon_0^{ab})$ represents pair creation at early times.

4.6 Determining $\langle\mathscr{N}\rangle$

In the following, we are interested in describing the emission in the i-th mode of particles that reach infinity at late times after propagating through the collapsing body. We are therefore interested in the part of the state vector $\Psi_i(\exp(-\pi\omega_i/\kappa)2_i\lambda^{(a}{}_i\tau^{b)}) \otimes \Psi_i(0)$ that belongs to $\mathscr{F}(\mathscr{H}_{out,\infty})$. Let us calculate the probability P_N for observing N particles at infinity in this mode in two cases.

(a) **$t_i = 1$ and $r_i = 0$, i.e. ${}_i\lambda^a = {}_i\rho^a$**
In this case, since exactly half of the particles produced by the black hole are emitted to infinity, P_N is simply proportional to the squared norm of the vector appearing in the $2N$-particle entry in the expression

$$\begin{aligned}\Psi_i(\exp(-\pi\omega_i/\kappa)2_i\lambda^{(a}{}_i\tau^{b)}) =&(1,0,2^{-1/2}\exp(-\pi\omega_i/\kappa)2_i\lambda^{(a}{}_i\tau^{b)},0,\\ &((3\cdot 1)/(4\cdot 2))^{1/2}\exp(-2\pi\omega_i/\kappa)4_i\lambda^{(a}{}_i\tau^{b}{}_i\lambda^{c}{}_i\tau^{d)},0,...).\end{aligned} \tag{109}$$

We obtain

$$\begin{aligned}P_N \propto& \tilde{P}_N = ((2N-1)(2N-3)...1)/(2N)(2N-2)...2)\\ &\cdot\exp(-N2\pi\omega/\kappa)2^{2N}\|{}_i\rho^{(a}{}_i\tau^{b}...{}_i\rho^{y}{}_i\tau^{z)}\|^2\\ &\overset{(1)}{=}((2N-1)(2N-3)...1)/(2N)(2N-2)...2)\\ &\cdot\exp(-N2\pi\omega/\kappa)2^{2N}(N!)^2/(2N)!\\ &\overset{(2)}{=}\exp(-N2\pi\omega/\kappa)\end{aligned}\quad . \tag{110}$$

Step (1) follows from the fact that

$${}_i\rho^{(a}{}_i\tau^{b}...{}_i\rho^{y}{}_i\tau^{z)} = \sqrt{\frac{1}{(2N)!}}\sum_{\sigma\in S_{2N}}{}_i\rho^{\sigma(a)}{}_i\tau^{\sigma(b)}...{}_i\rho^{\sigma(y)}{}_i\tau^{\sigma(z)} \tag{111}$$

where S_{2N} is the permutation group of a set of $2N$ elements and

$$\|{}_i\rho^{(a}{}_i\tau^{b}...{}_i\rho^{y}{}_i\tau^{z)}\|^2 = ({}_i\rho^{(a}{}_i\tau^{b}...{}_i\rho^{y}{}_i\tau^{z)},{}_i\rho^{(a}{}_i\tau^{b}...{}_i\rho^{y}{}_i\tau^{z)}) \tag{112}$$

is the scalar product defined in (18). Since the ${}_i\rho^a$ and the ${}_i\tau^a$ are orthogonal to each other and are normalized to 1 with respect to this scalar product, it follows that the only non zero contractions are those where the ${}_i\rho^a$ are contracted to each other ($N!$ possibilities) and the ${}_i\tau^a$ are contracted to each other ($N!$ possibilities). Step (2) follows from the fact that

$$\frac{(2N-1)(2N-3)...1)}{(2N)(2N-2)...2)} \cdot \frac{2^{2N}(N!)^2}{(2N)!} = 1. \tag{113}$$

If we now normalize the result of expression (110), we have that $P_N = \tilde{P}_N / (\sum_{M=0}^{\infty} \tilde{P}_M)$ and if we compute the expectation value of this distribution (using the fact that $\sum_{n=0}^{\infty} \exp(-an) = \frac{\exp(a)}{\exp(a)-1}$ and $\sum_{n=0}^{\infty} n \cdot \exp(-an) = \frac{\exp(a)}{(\exp(a)-1)^2}$, where $a = \frac{2\omega\pi}{\kappa}$), we obtain

$$\boxed{\langle \mathscr{N} \rangle = \sum_{N=0}^{\infty} N \cdot P_N = \frac{\exp(-2\pi\omega/\kappa)}{1-\exp(-2\pi\omega/\kappa)}.} \tag{114}$$

(b) $t_i \neq 1$, **i.e.** ${}_i\lambda^a = t_i\, {}_i\rho^a + r_i\, {}_i\sigma^a$

If $t_i \neq 1$, it sufficies to multiply the expression for P_N in the previous point by the probability $|t_i|^2$ for a particle to reach infinity. One finally obtains

$$\boxed{\langle \mathscr{N} \rangle = |t_i|^2 \frac{\exp(-2\pi\omega/\kappa)}{1-\exp(-2\pi\omega/\kappa)}.} \tag{115}$$

5 Discussion

We will first discuss our result and then try to give an interpretation of the phenomenon of particle creation.

Equation (115) gives the expected number of particles created in a given mode $i = jnlm$ and detected at the late retarded time $u \propto n$. As we have seen, at late times (when the balck hole appears to an observer at $\mathscr{I}^+$ to have settled down), the particle creation is only due to the scattering of the classical Klein-Gordon field through the collapsing body. Since the value of $\langle \mathscr{N} \rangle$ does not depend on n, particles are emitted at a finite steady rate for an infinite time. This will cause the black hole to lose mass. We also notice that equation (115) is precisely the rate of the thermal emission of a perfect blackbody at temperature given by

$$k_B T = \frac{\kappa}{2\pi},$$

where k_B is the Boltzmann constant. This can be interpreted as the temperature of the black hole.

An interpretation of the phenomenon of particle creation is given in [7]. Since, as we have shown, particles are created in pairs from the vacuum, we have

$$0 = P_1 + P_2, \tag{116}$$

where P_1 and P_2 are time-like vectors representing the four-momenta of the two particles. Then

$$0 = K_\mu P_1^\mu + K_\mu P_2^\mu = E_1 + E_2 \tag{117}$$

where $K = \partial/\partial t$ is the time traslation Killing vector and E_1 and E_2 represent the particles' energy. Since the Killing field outside the horizon is time-like, the two particles cannot be created outside the horizon, because then $E_1, E_2 > 0$, which would violate energy conservation.

If they are created inside the horizon, where Killing vector is space-like, E_1 and E_2 may have opposite signs, but they would never get outside. Therefore, one particle needs to be created inside and the other outside the horizon. This would be the physical reason of what we have mathematically shown above. Moreover, the particle of negative energy created inside the horizon could be interpreted as an antiparticle and could be responsible of the loss of mass via annihilation.

References

[1] S. W. Hawking, *Particle Creation by Black Holes.* Commun. Math. Phys. 43, 199-220, 1975.

[2] R. M. Wald, *On Particle Creation by Black Holes.* Commun. Math. Phys. 45, 9-34, 1975.

[3] A. Ghosh, *Hawking Radiation-Revisited.* arXiv: 1901.10069vl [gr-qc], 2019.

[4] R. M. Wald, *General Relativity.* The University of Chicago Press, 1984.

[5] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation.* W. H. Freeman and Company, 1973.

[6] S. Chandrasekhar, *The Mathematical Theory of Black Holes.* Oxford, Clarendon Press, 1983.

[7] G. M. Graf, *General Relativity HS14.* ETH Zurich, 2014.

[8] R. Renner, *General Relativity HS18.* ETH Zurich, 2018.

[9] P. Jetzer, *Advanced topics in general relativity and gravitational waves.* University of Zurich, 2017.

[10] Wikipedia, The Free Encyclopedia, *Schwarzschild metric* `https://en.wikipedia.org/wiki/Schwarzschild_metric`, 15.12.2019.

[11] Wikipedia, The Free Encyclopedia, *Kruskal-Szekeres coordinates* `https://en.wikipedia.org/wiki/Kruskal-Szekeres_coordinates`, 15.12.2019.

[12] Wikipedia, The Free Encyclopedia, *Surface gravity* `https://en.wikipedia.org/wiki/Surface_gravity`, 15.12.2019.

[13] `http://www.people.vcu.edu/~rgowdy/mod/104/imp.htm#2`, 15.12.2019.