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Post-Minkowski expansion of General Relativity

Semester Project

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Abstract

The aim of this report is to review in detail a recursive method to compute a *multipole post-Minkowski* expansion in general relativity as presented in [1] by S. Detweiler and L.H. Brown. The idea behind this work is to integrate the content of their paper with some additional information taken from previous literature so as to make the topic more accessible to the non-expert. Moreover, some of the calculations which were skipped in the paper are presented here.
A post-Minkowski approximation of general relativity is a power series expansion in the gravitational constant G. We will show how to compute a complete step in the iterative procedure, which takes an approximate solution of the Einstein's field equations and produces a new solution with an error decreased by a factor of G.
The choice of pursuing an alternative approach to the more popular post-Newtonian (PN) expansion is motivated by the fact that PN approaches are hampered by internal inconsistencies, due to the appearance of divergent integrals at higher orders in the approximation. Another advantage of this method is that nowhere we impose a particular gauge choice, in particular we do not require the existence of a harmonic coordinate system.

Notation and General Setting

We start by introducing some basic notation and conventions adopted throughout the report. The partial derivative is represented by the operator ∂_a and only the Minkowski metric η_{ab} and its inverse η^{ab} are used to lower and raise tensor indices. We will make extensive use of the d'Alambertian operator in Minkowski space time $\Box := \eta^{ab}\partial_a\partial_b$, while only occasionally we will use the covariant derivative of a tensor, denoted with ∇_a. The signature of the metric follows the mostly plus convention $(-+++)$, so that $\eta_{ab} = \eta^{ab} = diag(-1,+1,+1,+1)$.
Let us consider multiple sources of gravitational field. The position of each of them is described by a world line $z^a(s)$ parametrised by the proper time s. We will adopt outgoing-null spherical coordinates (s, r, θ, ϕ) centered on the world line. The field s at any event P is the value of s at the vertex of the future null cone through that point, i.e. $s(P) = s(Q)$ where Q is the vertex of the null cone from $z^a(s)$ containing P. Since P lies on the null cone of Q, the space time interval between x^a and z^a is

$$\Omega(x^a, z^a(s)) = \eta_{ab}(x^a - z^a(s))(x^b - z^b(s)) = 0. \tag{2.1}$$

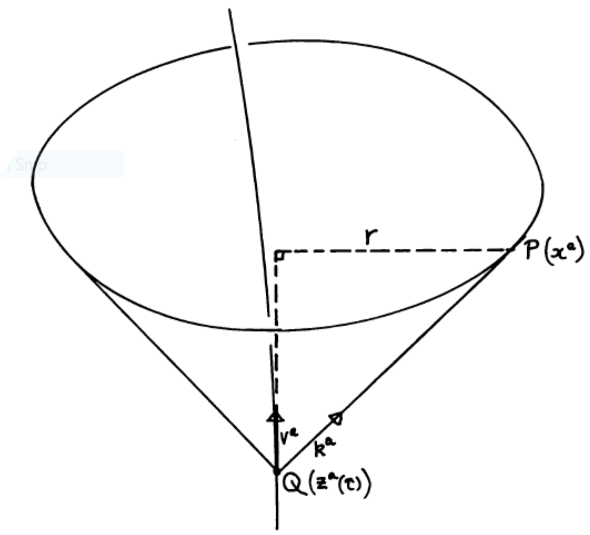

Figure 2.1: Outgoing-null spherical coordinates. P is on the future null cone of $Q(z^a)$ and k^a is a null vector along the distance of the two points.

Following [2], let r be the spatial distance in Minkowski spacetime between Q and P as measured in the instaneous rest frame of Q at the appropriate advanced time. The setting is depicted in Figure 2.1. Then, in the rest frame of Q, where $v = (-1, 0, 0, 0)$, $r = \left|x^0 - z^0(s)\right| = x^0 - z^0(s)$ because $r \geq 0$. Therefore, moving to a generic reference frame the radial coordinate is equal to

$$r(x^a) = -v_a[x^a - z^a(\mathcal{Q})]. \tag{2.2}$$

In terms of the null spacetime separation vector, we can also define

$$k^a(x) := \frac{x^a - z^a(Q)}{r} \tag{2.3}$$

and notice that $k^a v_a = -1$. Finally, θ and ϕ are the usual angles as defined in the standard spherical coordinates. Because P and Q are always on each other's null cones, if the world line $z^a(s)$ is given, then Q is determined uniquely by P, as it is located on the intersection of the world line and the past null cone of P.

Taking the derivative of a tensor field T^{ab} defined along a world line is an easy task by use of the chain rule:

$$\partial_c T^{ab} = -(\partial_c s)\frac{\partial T^{ab}}{\partial s} = -k_c \frac{\partial T^{ab}}{\partial s}, \tag{2.4}$$

where in the second equality we used that $0 = -ds + k_a dx^a$ so $ds/dx^b = k_b$. We can derive some basic relations which will be useful throughout the rest of the analysis:

$$\begin{aligned}
\partial_b r(x^a) &= \partial_b(-v_a x^a) + v_a z^a(Q)) \\
&= -v_a(\partial_b x^a - \partial_z^a(Q)) - \partial_b v_a(x^a - z^a(Q)) \\
&= -v_a(\delta_b^a + k_b v^a) + k_b \dot{v}_a r k^a \\
&= -v_b + k_b(1 + r k_a \dot{v}^a) \\
&= n_b + r k_b k_a \dot{v}^a,
\end{aligned} \tag{2.5}$$

where $n_a := k_a - v_a$ is an outward-pointing spatial unit vector. Using the relation just found and the Leibniz rule of the derivative,

$$\begin{aligned}
r\partial_a k_b &= \partial_a(r k_b) - (\partial_a r)k_b \\
&= \partial_a(x_b - z_b(Q)) - (-v_a + k_a + r k_a k_c \dot{v}^c)k_b \\
&= \partial_a(x^c \eta_{bc}) + k_a v_b + v_a k_b - k_a k_b(1 + r k_c \dot{v}^c) \\
&= \eta_{ab} + v_a k_b + v_b k_a - k_a k_b(1 + r k_c \dot{v}^c).
\end{aligned} \tag{2.6}$$

Let us now introduce the metric g_{ab} of a generic space time as a symmetric and invertible tensor. It is convenient to split it in a background Minkowski metric plus a perturbation and define h^{ab} in terms of the inverse metric g^{ab} by

$$\mathfrak{g}^{ab} = \sqrt{-g}g^{ab} := \eta^{ab} - h^{ab}. \tag{2.7}$$

We also introduce the Einstein tensor density

$$E^{ab}(h) = (-2g)(R^{ab} - \frac{1}{2}R g^{ab}), \tag{2.8}$$

which is explicitly related to h^{ab} through the following relation, as given by [3]:

$$E^{ab}(h) = -\Box h^{ab} + \partial^a \partial_c h^{cb} + \partial^b \partial_c h^{ca} - \eta^{ab}\partial_c \partial_d h^{cd} - \tau^{ab}(h). \tag{2.9}$$

The quantity $\tau^{ab}(h)$ contains all the quadratic terms in h and its derivatives. The Bianchi identity rewritten in terms of the Einstein tensor density E^{ab} is easily computed:

$$
\begin{aligned}
\nabla_a(G^{ab}) &= \nabla_a\left(\frac{1}{2(-g)}E^{ab}\right) \\
&= -\frac{1}{2(-g^2)}gg^{de}(\partial_a g_{de})E^{ab} + \frac{1}{2(-g)}\partial_a E^{ab} + \frac{1}{2(-g)}\Gamma^c_{ca}E^{ab} + \frac{1}{2(-g)}\Gamma^b_{ac}E^{ac} \\
&= \frac{1}{2(-g)}\bigg(- g^{de}(\partial_a g_{de})E^{ab} + \partial_a E^{ab} + \Gamma^c_{ca}E^{ab} + \Gamma^b_{ac}E^{ac}\bigg) \\
&= \frac{1}{2(-g)}\bigg(- 2\Gamma^c_{ca}E^{ab} + \partial_a E^{ab} + \Gamma^c_{ca}E^{ab} + \Gamma^b_{ac}E^{ac}\bigg) = 0.
\end{aligned}
\tag{2.10}
$$

To go from the third to the fourth line we used the fact that

$$
E^{ab}\Gamma^d_{da} = \frac{1}{2}g^{de}(\partial_a g_{de} + \partial_d g_{ea} - \partial_e g_{da})E^{ab} = \frac{1}{2}g^{de}(\partial_a g_{de})E^{ab}, \tag{2.11}
$$

where the last two terms in parenthesis are antisymmetric in the indices d and e and vanish when they are contracted with the inverse metric g^{de}. Finally we can express the divergence of E^{ab} as

$$
\begin{aligned}
\partial_a E^{ab} &= 2\Gamma^c_{ca}E^{ab} - \Gamma^c_{ca}E^{ab} - \Gamma^b_{ac}E^{ac} \\
&= (\eta_c{}^b\,\Gamma^d_{da} - \Gamma^b_{ac})E^{ac}.
\end{aligned}
\tag{2.12}
$$

Let us turn our focus on the post-Minkowski expansion. The iterative procedure is based on a "nonlinearity expansion" of the metric tensor in powers of a parameter ϵ:

$$
h_{ab}(\epsilon) = \sum_n \epsilon^n h^{(n)}_{ab}. \tag{2.13}
$$

As it is noted in [4], the parameter ϵ does not have any physical significance, and this series does not represent the physical field. The only role of ϵ is to attribute a weight to every term of the series so as to define their order of magnitude and give a meaning to the expansion. The idea of the post-Minkowski approach is to adopt $\epsilon \equiv G$ and assume that it is a small quantity compared to the characteristic masses and distances of the problem at hand. The power of this method resides in the fact that we are not imposing any gauge condition on the metric tensor, clearly in contrast to the commonly adopted restriction to the harmonic gauge $\partial_b(\sqrt{-g}g^{ab}) = \Gamma^{cd}_{\ a}g_{cd} = 0$. This represents an advantage, as for systems with strong internal gravity, e.g. black holes, a harmonic coordinate system may not be suited to cover the whole manifold in a nonsingular manner (see [5]).

Let us implement this expansion for the setting presented above. At order zero $h^{ab}_0 = 0$, at first order $h^{ab}_1 = \mathcal{O}(G)$, and so forth. In general

$$
\mathfrak{g}_{ab} = \sqrt{-g}\, g_{ab} = \eta_{ab} + \sum_n G^n h_{ab\,(n)}. \tag{2.14}
$$

As it is not possible to expand the field at a singularity, we enclose every source by boundaries. In spacetime these boundaries are 3-dimensional surfaces of constant radius in outgoing-null spherical coordinates centered around each source. They have the topological properties of time-like cylinders. In this way we avoid to introduce an energy-momentum tensor and the geometric data is given as conditions on the boundaries.

Outside the inner boundaries the metric $\mathfrak{g}_a b$ satisfies the Einstein's equations in the vacuum, which in terms of 2.8 are

$$E^{ab}(h) = 0. \tag{2.15}$$

This means that at every order the metric $h^{ab}_{(n)}$ satisfies (see [6])

$$\begin{aligned} &\partial_a h^{ab}_{(n)} = H^b_{(n)} \\ &\Box h^{ab}_{(n)} = \partial H^{ab}_{(n)} + N^{ab}_{(n)}(h^{ab}_{(m)}) \qquad \text{with } m < n. \end{aligned} \tag{2.16}$$

Here $N^{ab}_{(n)}$ is a source term depending on $h^{ab}_{(m)}$ and $\partial H^{ab}_{(n)}$ is a gauge term which is $\mathcal{O}(G^{n+1})$ and has the form

$$\partial H^{ab}_{(n)} = \partial^a \lambda^b_{(n)} + \partial^b \lambda^a_{(n)} - \eta^{ab}\partial_c \lambda^c_{(n)}. \tag{2.17}$$

From now on we will drop the parenthesis on the indexes which indicate the order of the approximation. At every order we assume that

$$E^{ab}(h_n) = \mathcal{O}\left(G^{n+1}\right), \tag{2.18}$$

and at every iterative step we compute a correction $\delta h^{ab}_n = \mathcal{O}\left(G^n\right)$ defined by

$$h^{ab}_n(x) \coloneqq h^{ab}_{n-1}(x) + \delta h^{ab}_n(x;G). \tag{2.19}$$

The improved approximation is given by a solution of the following differential equation:

$$\begin{aligned} \Box \delta h^{ab}_n &= \Box(h^{ab}_n - h^{ab}_{n-1}) \\ &= -E^{ab}(h^{ab}_n) + \partial^a\partial_c h^{cb}_n + \partial^b\partial_c h^{ca}_n - \eta^{ab}\partial_c a \partial_d h^{cd}_n - 16\pi\tau^a b(h_n) + \\ &\quad + E^{ab}(h^{ab}_{n-1}) - (\partial^a\partial_c h^{cb}_{n-1} + \partial^b\partial_c h^{ca}_{n-1} - \eta^{ab}\partial_c a\partial_d h^{cd}_{n-1} - 16\pi\tau^a b(h_{n-1})) \\ &= E^{ab}(h^{ab}_{n-1}) - E^{ab}(h^{ab}_n) + \partial^a\partial_c\delta h^{cb}_n + \partial^b\partial_c\delta h^{ca}_n - \eta^{ab}\partial_c\partial_d\delta h^{cd}_n - 16\pi\tau^{ab}(\delta h_n) \\ &= E^{ab}(h^{ab}_{n-1}) + \mathcal{O}\left(G^{n+1}\right). \end{aligned} \tag{2.20}$$

In the last line we used the recursive condition 2.18. Note that τ^{ab} is quadratic in the metric and its derivatives, so it does not contain terms $\mathcal{O}(G^n)$. Furthermore, in order to write the final expression we also require the restriction

$$\partial_a \delta h^{ab}_n = \mathcal{O}(G^{n+1}). \tag{2.21}$$

We verify that 2.19 is a more accurate solution of the field equations by plugging the solution into Einstein's tensor density and by adding and subtracting the same quantity $E^{ab}(h_{n-1})$,

$$\begin{aligned} E^{ab}(h_n) &= E^{ab}(h_{n-1} + \delta h_n) - E^{ab}(h_{n-1}) + \Box\delta h^{ab}_n + \mathcal{O}(G^{n+1}) \\ &= \partial^a\partial_c\delta h^{cb}_n + \partial^b\partial_c\delta h^{ca}_n - \eta^{ab}\partial_c\partial_d\delta h^{cd}_n - \Big(\tau^{ab}(h_{n-1} + \delta h_n) - \tau^{ab}(h_{n-1})\Big) + \mathcal{O}(G^{n+1}) \\ &= \mathcal{O}(G^{n+1}). \end{aligned} \tag{2.22}$$

In the first line we used 2.20 and in the second we used 2.9 together with the linearity of 2.9 in h^{ab} for the first four terms. Using 2.21 we conclude that the first three terms are $\mathcal{O}(G^{n+1})$, as

well as the quadratic term in brackets, since each of its terms contains at least a first derivative of the metric.
The other main ingredient of this expansion is the decomposition into tensor multipole moments. To make use of it we need to introduce some additional elements. First, we will shorten the notation for the tensor product of l unit vectors n^a (cf. 2.5) by writing it as $N^L := n^{a_1}...n^{a_l}$, with the capital superscript letter indicating the number of indices in the sequence. In a similar way, the sequence of l partial derivatives will be denoted as $\partial_L := \partial_{a_1}...\partial_{a_l}$. Secondly, we will represent the symmetric part of a tensor with parenthesis on its indices

$$T_{(abc)de} := \frac{1}{3!}(T_{abcde} + T_{acbde} + T_{bacde} + T_{bcade} + T_{cabde} + T_{cbade}). \tag{2.23}$$

We will make extensive use from now on of tensors which are symmetric and completely trace-free (STF). The STF part of a tensor is obtained in two steps. First we take the symmetric part

$$S_{a_1..a_N} = \frac{1}{l!}\sum_{\pi} A_{a_{\pi(1)}...a_{\pi(N)}}. \tag{2.24}$$

The summation goes over the $l!$ permutations π of the indices. Then we remove all the traces

$$[T_{abc}]^{STF} = T_{(abc)} - \frac{1}{5}\left[\delta_{ab}T_{(kkc)} - \delta_{ac}T_{(kbk)} - \delta_{bc}T_{(akk)}\right]. \tag{2.25}$$

It is easy to verify that the factor in front is chosen so that this tensor vanishes when contracting two of the three free indices, or in other words, so that it is indeed traceless. However, the simplicity of this example can be misleading, for we might add just one index and the procedure above would not work anymore. Indeed, let us proceed as before and assume a form like

$$[T_{abcd}]^{STF} = T_{(abcd)} - a\,\delta_{(ab}T_{cd)e}{}^{e}. \tag{2.26}$$

With the parameter a to be fixed. In particular this does not work because the trace of the right hand side contains a term of the form $\frac{1}{6}a\delta_{cd}T_{bbee}$ which cannot be set to zero by any choice of $a \neq 0$. That is why it is necessary to assume the more general form

$$[T_{abcd}]^{STF} = T_{(abcd)} - a\,\delta_{(ab}T_{cd)e}{}^{e} + b\,\delta_{(ab}\delta_{cd)}T_{b}{}^{b}{}_{e}{}^{e} \tag{2.27}$$

Following [2], this suggests that in general we should assume an expression of the form

$$\begin{aligned}
[T_{a_1...a_N}]^{STF} &= T_{a_1...a_N} - a_2\,\delta_{(a_1a_2}T_{a_3a_N)b}{}^{b} + a_4\,\delta_{(a_1a_2}\delta_{a_3a_4}T_{a_5...a_N)b_1}{}^{b_1}{}_{b_2}{}^{b_2} + ... \\
&= \sum_{k=0}^{[\frac{N}{2}]}(-1)^k a_{2k}\,\delta_{(a_1a_2}...\delta_{a_{2k-1}a_{2k}}T_{a_{2k+1}a_N)b_1}{}^{b_1}...{}_{b_N}{}^{b_N}.
\end{aligned} \tag{2.28}$$

Here $[\frac{N}{2}]$ indicates the largest integer less than or equal to $\frac{N}{2}$. Allowing for the symmetrisation of the indices, a contraction of the k-th term gives a term which is contracted k times, or $k+1$ times. Therefore, when cancelling terms after a contraction we have to deal only with the k-th and $(k+1)$-th terms. Choosing two indices a_1 and a_2 for the contraction, and counting the possible ways in which these contractions can be placed among the $N!$ terms of the right hand side of 2.28 we come up with the following condition for the prefactor:

$$a_{2(k+1)} = \frac{(N-2k)(N-2k-1)}{2(k+1)(2N-2k-1)}a_{2k}, \tag{2.29}$$

which can be solved by induction, with the base case $a_0 = 1$,

$$
\begin{aligned}
a_{2k} &= \frac{\binom{N}{k}\binom{N}{2k}}{\binom{2N}{2k}} = \frac{N!}{k!(N-k)!}\frac{N!}{2k!(N-2k)!}\frac{2k!(2N-2k)!}{(2N)!} \\
&= \frac{N!}{k!(N-k)!}\frac{N!}{(N-2k)!}\frac{(2N-2k)!!(2N-2k-1)!!}{(2N)!!(2N-1)!!} = \frac{(2N-2k-1)!!N!}{k!(N-2k)!(2N-1)!!}.
\end{aligned}
\tag{2.30}
$$

As a reminder, the semifactorial of n is defined as $n!! = n(n-2)(n-4)...$ and is related to the factorial by the property $n! = n!!(n-1)!!$.
Just like the spherical harmonics $Y(\theta,\phi)$, the set of all STF tensors of rank l generates an irreducible representation of the (spatial) rotation group $O(3)$ of weight l. Indeed there exists a bilinear mapping between them and the spherical harmonics (cf. [5]). Let us consider a generic scalar function $f(\theta,\phi)$; we can expand it as a power series in the unit radial vector n with coefficients that are rank-l STF tensors,

$$
f(\theta,\phi) = \sum_L F_L N_L(\theta,\phi). \tag{2.31}
$$

We will also impose that these STF tensors be spatial with respect to $v^a(s)$. This is obtained by means of a projection operator

$$
f^{ab} := \eta^{ab} + v^a v^b, \tag{2.32}
$$

which projects every tensor to a spatial three manifold instantaneously orthogonal to v^a. The symmetric and trace-free tensors which are also spatial are dubbed "SSTF" for short. Moreover, the totally antisymmetric tensor which is orthogonal to v^a is

$$
\epsilon^{abc} := \epsilon^{abcd} v_d, \tag{2.33}
$$

of which we will make extensive use in the decomposition of the metric tensor and its derivatives. From now on we will reserve capital script letters like $\mathcal{A}$ or $\mathcal{P}$ to denote SSTF tensors which depend on s but are independent of θ, ϕ and r.

Going back to our expansion scheme, we can expand the metric tensor at every order as a power series in a similar way,

$$
h^{ab\,(n)}(\mathbf{x},t) = \sum_L N^L(\theta,\phi) h_L^{ab\,(n)}(r,t). \tag{2.34}
$$

To obtain the radiation field we need to take the transverse-traceless part of the most general metric given by 2.34.
This kind of simultaneous expansion is expected a priori to produce a good approximation everywhere outside the source, assuming that the expansion is indeed convergent, as remarked in [6]. This is different, for example, from an expansion in inverse powers of r at fixed null coordinate, which produces a valid approximation only asymptotically. To be more precise we can characterise a source of gravitational waves by some length scales: its Schwarzschild radius $2GM/c^2$ ($= 2M$ with $G = c = 1$) and the characteristic wavelength of its gravitational waves. According to these scales we can consider some regions in the space around the source: a weak field zone, characterised by $r \gg \frac{GM}{c^2}$ and a far wave zone, characterised by $r \gg \lambda$. The nonlinearity approach followed here is expected to be valid in the former, assuming that the radius of the source is smaller than its Schwarzschild radius, while the asymptotic expansion will yield a good approximation only in the latter.

Linear order approximation

The scope of this section is to compute a linear solution of the field equations in the post-Minkowski expansion scheme. We consider now the retarded Green's function $G^{+}(x-x') \equiv G(x-x')$ of the d'Alembert operator in flat spacetime, which is a solution of

$$\Box G(x-x') = -4\pi\delta^{4}(x-x'). \tag{3.1}$$

It can be easily checked that $G(x-x') = 2\theta\big(x^{0}-x'^{\,0}\big)\delta\big(\Omega(x-x')\big)$, where Ω is the spacetime interval as defined in 2.1. We can rewrite this result in a more useful form by expanding $\delta(\Omega)$:

$$\begin{aligned}
\delta(\Omega) &= \delta\big(\eta_{ab}(x^{a}-x'^{a})(x^{b}-x'^{b})\big)\\
&= \delta\big([(x^{0}-x'^{0})+(x^{i}-x'^{i})][-(x_{0}-x'_{0})+(x_{i}-x'_{i})]\big)\\
&= \delta\big(-(x^{0}-x'^{0})^{2}+(x^{i}-x'^{i})^{2}\big)\\
&= \delta\big(-(s_{x}-s)^{2}+(r)^{2}\big)\\
&= \delta\big(r^{2}-(s_{x}-s)^{2})\big)\\
&= \frac{\delta\big(s-(s_{x}-s)\big)}{2|s-s_{x}|}\bigg|_{s>s_{x}} + \frac{\delta\big(s-(s-s_{x})\big)}{2|s-s_{x}|}\bigg|_{s<s_{x}}\\
&= \frac{\delta\big(s_{x}\big)}{2r}\bigg|_{s>s_{x}} + \frac{\delta\big(2s-s_{x}\big)}{2r}\bigg|_{s<s_{x}}.
\end{aligned} \tag{3.2}$$

In the fourth line we moved to a outgoing-null coordinate system and from the fifth to the sixth line we used the well known property of the delta function. In the retarded solution the second term drops, since $s > s_{x}$, and we obtain

$$G(x-z(s)) = \frac{1}{r}\delta(s_{x})\theta(s-s_{x}) \tag{3.3}$$

As stated in the previous section, we are concerned with the inhomogeneous wave equation

$$\Box h^{ab} = -\rho^{ab}, \tag{3.4}$$

where ρ^{ab} can be expressed in terms of its multipolar decomposition, also called multipolar skeleton,

$$\rho^{ab}(x) = 4\pi\int \mathcal{M}_{1}^{abL}\partial_{L}\delta^{(4)}\big(x-z(s)\big)ds. \tag{3.5}$$

The solution is given by the convolution of the Green's function 3.3 with ρ^{ab},

$$
\begin{aligned}
h^{ab}(x) &= \int d^4x'\, G(x-x')\rho^{ab}(x') \\
&= \int d^4x'\, G(x-x')\, \partial'_L \int ds\, \mathcal{M}_1^{abL}(s)\delta^{(4)}(x'-z_s) \\
&= -\int d^4x' \Big[\partial'_{d_1} G(x-x')\, \partial'_{L(\hat{l}_1)} \int ds\, \mathcal{M}_1^{abL}(s)\delta^{(4)}(x'-z_s) + \\
&\quad + \partial'_L \int ds\, G(x-x')\mathcal{M}_1^{abL}(s)\delta^{(4)}(x'-z_s)\Big] \\
&= -\int d^4x'\, \partial'_{d_1} G(x-x')\, \partial'_{L(\hat{l}_1)} \int ds\, \mathcal{M}_1^{abL}(s)\delta^{(4)}(x'-z_s) \\
&= -\int ds\, \frac{\partial}{\partial z_s} \partial'_{L(\hat{l}_1)} G(x-z_s)\mathcal{M}_1^{abL}(s) \\
&= \partial_{d_1} \int ds\, \partial'_{L(\hat{l}_1)} G(x-z_s)\mathcal{M}_1^{abL}(s) \\
&= \text{...(repeat integration by part l times)...} \\
&= \partial_L \int ds\, G(x-z_s)\mathcal{M}_1^{abL}(s) \\
&= \partial_L \int ds\, \frac{1}{r}\delta(s_x)\theta(s-s_x)\mathcal{M}_1^{abL}(s) \\
&= \partial_L \left[\frac{1}{r}\mathcal{M}_1^{abL}(s_x)\right].
\end{aligned}
\tag{3.6}
$$

The relation just obtained is important for the rest of the discussion. Here the notation used for the partial derivatives is to be understood as $\partial_{L(\hat{l}_k)} = \partial_{d_1}...\partial_{d_{k-1}}\partial_{d_{k+1}}...\partial_{d_l}$. On the third line we integrated by parts on the variable x'_{l_1} and then dropped the boundary term on the fourth, by inverting the order of the integrals and knowing that $G(x-z(s))$ vanishes far away from the source. Secondly, after realising that the integrand depends on $z(s)$ only through the Green's function in the form $x-z(s)$ (fifth line), we replaced the derivative over $z(s)$ with minus the derivative over x_{l_1} and we pulled it out of the integral. This steps had to be repeated l times in order to get to the final expression.
The result of 3.6 is the most general metric field which satisfies the wave equation 3.4 and which contains only outgoing waves. Incidentally, this last point depends on the definition of the null coordinate we gave at the beginning. If we were interested in incoming wave solutions, we could have defined s at a point P as $s(P) = s(Q)$ where Q is the vertex of the *past* null cone containing P. We express 3.6 as a sum over its multipole components:

$$
\begin{aligned}
\mathcal{M}_1^{abL} &= v^a v^b \mathcal{A}_1^L + 2v^{(a}\mathcal{B}_1^{b)L} + 2v^{(a}\epsilon_q^{b)(d_l}\mathcal{C}_1^{L-1)q} + 2v^{(a}f^{b)(d_l}\mathcal{D}_1^{L-1)} + f^{ab}\mathcal{E}_1^L + \mathcal{F}_1^{abL} + \\
&\quad + 2\epsilon_q^{a)(d_l}\mathcal{G}_1^{L-1)q(b} + 2f^{a)(d_l}\mathcal{H}_1^{L-1)(b} + 2\epsilon^{a)q(d_l}\mathcal{J}_{1q}^{L-2}f^{d_{l-1})(b} + f^{a(d_l}\mathcal{K}_1^{L-2}f^{d_{l-1})b}.
\end{aligned}
\tag{3.7}
$$

Here the reverse parenthesis serve to represent the symmetrisation over the indices a and b. If the notation seems chaotic, it is good to keep in mind that only indices pertaining to the same SSTF tensor can be symmetrised, as every SSTF mode is independent from the others. We remark that the multipoles $\mathcal{B}_1^L$,$\mathcal{C}_1^L$,$\mathcal{H}_1^L$ and $\mathcal{J}_1^L$ lack the monopole mode $l=0$, while $\mathcal{F}_1^L$ and $\mathcal{G}_1^L$ do not even have a dipole mode $l=1$. Moreover, $\mathcal{F}_1^L$ and $\mathcal{G}_1^L$ both give a contribution to h_{ab}^1 which is transverse to n^a and traceless. This means that they represent the moments of the sources of gravitational waves located inside the inner boundaries. Even though we are not imposing here any gauge condition, a restriction on these SSTF objects arises as a consequence of 2.21.

N-th order approximation

The content of this section is the core of this expansion, as it provides a way to get from an approximate solution of the field equations to a solution of a higher order, more accurate by a factor of G. The idea is to start from the linear solution just computed and apply the algorithm recursively.
In order to do so, we start by considering the correction of the metric at the n-th order δh_n^{ab} as a sum of two terms:

$$\delta h_n^{ab} = p_n^{ab} + q_n^{ab}. \tag{4.1}$$

We define q_n^{ab} to satisfy the homogeneous wave equation

$$\Box q_n^{ab} = 0. \tag{4.2}$$

The other term p_n by definition is a solution of the inhomogeneous equation

$$\Box p_n^{ab} = E_{n-1}^{ab}. \tag{4.3}$$

Because p_n^{ab} can be decomposed in a unique way into SSTF tensors we can algorithmically construct q_n^{ab} (cf. [6]), with the additional condition on their derivatives that

$$\partial_a p_n^{ab} + \partial_a q_n^{ab} = \mathcal{O}(G^{n+1}). \tag{4.4}$$

This is an important point, as it will allow later to find a relation between the SSTF objects constituting the two components of the metric.
Being a solution of the homogeneous wave equation, q_n^{ab} is given by a decomposition into multipole moments which is analogous to the one constructed for the metric at the first order (cf. 3.7):

$$q_n^{ab}(x) = \sum_{l=0}^{\infty} \partial_L \left[\frac{1}{r} M_n^{abL}(s_x) \right]. \tag{4.5}$$

We note that this satisfies 4.2 everywhere except on the world line, as it is clearly singular for $r = 0$. We will then modify the definition 4.2 accordingly, to avoid this problem.
As before, a formal solution is given by the convolution of the Green's function of the $\Box$ operator with the nonhomogeneous term,

$$p_n^{ab} = -\frac{1}{4\pi} \int E_{n-1}^{ab}(x') G(x - x') d^4 x'. \tag{4.6}$$

The derivative acts only on the Green's function and therefore

$$
\begin{aligned}
\partial_a p_n^{ab} &= -\frac{1}{4\pi}\int d^4x' E_{n-1}^{ab}(x')\partial_a G(x-x') \\
&= \frac{1}{4\pi}\int d^4x' E_{n-1}^{ab}(x')\partial'_a G(x-x') \\
&= \frac{1}{4\pi}\int d^4x' \partial'_a[E_{n-1}^{ab}(x')G(x-x')] - \frac{1}{4\pi}\int d^4x' \partial'_a E_{n-1}^{ab}(x')G(x-x') \\
&= \frac{1}{4\pi}\int d^4x' \partial'_a[E_{n-1}^{ab}(x')G(x-x')] + \mathcal{O}(G^{n+1}) \\
&= -\frac{1}{4\pi}\int ds'\left[\oint \partial'_a r' E_{n-1}^{ab}(x')G(x-x')r_0^2\sin\theta' d\theta' d\phi'\right] + \mathcal{O}(G^{n+1}).
\end{aligned}
\tag{4.7}
$$

We made use of the antisymmetry of the Green's function to replace the derivative over x with the derivative over x' which allowed us then to integrate by parts. Once more we used 2.12 to ascribe the second term in the third line to a term of higher order in G, as the Christoffel symbols are $\mathcal{O}(G)$. In the last passage we wrote the integration over the spatial coordinates explicitly and applied Stokes' theorem, integrating over the inner boundaries at constant radius (the radial vector is normal). The minus sign is due to the convention chosen for the orientation of the boundaries.

Let us consider a generic point x^a. It is clear from 3.3 that the Green's function is nonvanishing only on its past null cone. Consequently, the integral vanishes always except for the values of s such that the boundaries intersect the past null cone, that is, at s and at about $s-2r_0$. This idea is represented graphically in Figure 4.1.

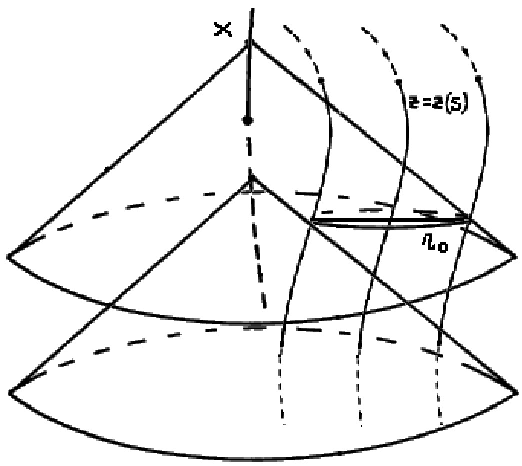

Figure 4.1: Minkowski diagram of the intersection of the past null cone with an inner boundary. The point of intersection closest to the field point has null coordinate value s and the one further away has value approximately $s-2r_0$ for a spherical boundary of radius r_0.

Now we want to make a Taylor expansion of the Green's function into multipole modes and we choose to do it around $s-r_o$, the most convenient point in light of what we have just observed,

$$
G(x^a - x'^a) = \sum_{l=0}^{\infty}\frac{(-r')^l}{l!}N'^L\partial_L G[x^a - z^a(s-r_0+r')] + \mathcal{O}(G). \tag{4.8}
$$

Plugging it into 4.7 and using that $\partial'_a r'\big|_{s-r_0} = \partial'_a r'\big|_s + \mathcal{O}(G)$,

$$\begin{aligned}
\partial_a p_n^{ab} &= -\frac{1}{4\pi}\sum_{l=0}^{\infty}\int ds'\left[\frac{(-r_0)^l}{l!}\oint \partial'_a r' E_{n-1}^{ab}(s'-r_0)N'^L\partial_L G(x-z(s'))r_0^2\sin\theta' d\theta' d\phi'\right] + \mathcal{O}(G^{n+1})\\
&= -\sum_{l=0}^{\infty}\int ds'\left[\frac{(-r_0)^{l+2}}{4\pi l!}\oint \partial'_a r' E_{n-1}^{ab}(s'-r_0)N'^L\partial_L G(x-z(s'))\sin\theta' d\theta' d\phi'\right] + \mathcal{O}(G^{n+1})\\
&= -\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\int ds'\,\frac{(-r_0)^{l+2m+2}}{4\pi(l+2m)!}\frac{d^{2m}}{ds'^{2m}}\left[\oint \partial'_a r' E_{n-1}^{ab}(s'-r_0)\hat{N}'^L\sin\theta' d\theta' d\phi'\right]\times\\
&\quad\times\partial_L G(x-z(s')) + \mathcal{O}(G^{n+1})\\
&= -\sum_{l=0}^{\infty}\int ds'\,\mathcal{K}_n^{bL}(s)\,\partial_L G(x-z(s)) + \mathcal{O}(G^{n+1}).
\end{aligned} \tag{4.9}$$

To go from the second to the third line we expressed N^L in terms of the combination of SSTF tensors $\hat{N}^L$, as our aim is to express the integrand in terms of SSTF tensors. To achieve that, we applied 2.32 as described in the introduction and then we applied the identity: $\int ds f(s)v^a\partial_a G(x-z(s)) = \int ds \frac{df}{ds}G(x-z(s))$ for $2m$ consecutive times to transform the term $N'^L\partial_L G(x-z(s))$. At the end we grouped the part integrated along the angular coordinates along with the prefactors and the summation over m in the term $K_n^{bL}(s)$ and we express it in terms of SSTF tensors:

$$\mathcal{K}_n^{bL}(s) = v^b\mathcal{P}_n^L(s) + \epsilon^b{}_{q\langle d_l}\mathcal{Q}_n^{L-1\rangle q}(s) + \mathcal{R}_n^{bL}(s) + f^{\,b\langle d_l}\mathcal{S}_n^{L-1\rangle}(s). \tag{4.10}$$

Here by $\langle\cdot\rangle$ we indicate the SSTF part of a tensor. After some effort we have finally obtained a decomposition of $\partial_a p_n^{ab}$. The tensors $\mathcal{P}^L\ldots\mathcal{S}^L$ determine the structure of the sources. Thanks to 4.4 we can relate the SSTF components of the two parts and obtain in particular the following matching conditions at every order:

$$\begin{aligned}
&\dot{\mathcal{A}}_n^L + \mathcal{B}_n^L + \ddot{\mathcal{D}}_n^L - \mathcal{P}_n^L = \mathcal{O}(G^{n+1})\\
&\dot{\mathcal{B}}_n^L + \mathcal{F}_n^L + \ddot{\mathcal{H}}_n^L - \mathcal{R}_n^L = \mathcal{O}(G^{n+1})\\
&\dot{\mathcal{C}}_n^L + \mathcal{G}_n^L + \ddot{\mathcal{J}}_n^L - \mathcal{Q}_n^L = \mathcal{O}(G^{n+1})\\
&\dot{\mathcal{D}}_n^L + \mathcal{E}_n^L + \mathcal{H}_n^L + \ddot{\mathcal{K}}_n^L - \mathcal{S}_n^L = \mathcal{O}(G^{n+1}).
\end{aligned} \tag{4.11}$$

This set of coupled differential equation is fundamental, as any solution to them provides a corresponding q_n^{ab} and together with p_n^{ab} completes the task of obtaining an improved solution of the field equations. We note that every equation is linear and homogeneous so the system can be solved by standard methods without much computational effort.

However, this alone is not enough to complete successfully a recursive step. There is indeed an aspect to consider regarding the solution of 4.11 which can be illustrated by a heuristic reasoning. The consequences of this observation call for an additional step in the recursive procedure and will be described in the next section.
The equations 4.11 can be seen as a set of inhomogeneous differential equations for the modes $\mathcal{A}_n^L\ldots\ \mathcal{K}_n^L$. As it is well known the general solution is given by the solution of the homogeneous equation plus a particular solution of the inhomogeneous equation. A possibility for the latter is to consider the system

$$
\begin{aligned}
\mathcal{B}_n^L &= \mathcal{P}_n^L + \mathcal{O}(G^{n+1}) \\
\mathcal{F}_n^L &= \mathcal{R}_n^L - \dot{\mathcal{B}}_n^L + \mathcal{O}(G^{n+1}) \\
\mathcal{G}_n^L &= \mathcal{Q}_n^L + \mathcal{O}(G^{n+1}) \\
\mathcal{E}_n^L &= \mathcal{S}_n^L + \mathcal{O}(G^{n+1}),
\end{aligned}
\tag{4.12}
$$

Obtained from the complete one by setting the remaining multipoles to zero. This is a sensible choice from a physical point of view, as it can describe steady objects which are not affected by tidal forces. Such a solution however fails for $l = 0$ and $l = 1$ as the low modes $\mathcal{B}_n$, $\mathcal{F}_n^a$ and $\mathcal{G}_n^a$ do not exist. This means we have to take into account the pathological cases as well,

$$
\begin{aligned}
&\dot{\mathcal{A}}_n + \ddot{\mathcal{D}}_n - \mathcal{P}_n = \mathcal{O}(G^{n+1}) \\
&\dot{\mathcal{A}}_n^b + \mathcal{B}_n^b + \ddot{\mathcal{D}}_n^b - \mathcal{P}_n^b = \mathcal{O}(G^{n+1}) \\
&\dot{\mathcal{B}}_n^b + \ddot{\mathcal{H}}_n^b - \mathcal{R}_n^b = \mathcal{O}(G^{n+1}) \\
&\dot{\mathcal{C}}_n^b + \ddot{\mathcal{J}}_n^b - \mathcal{Q}_n^b = \mathcal{O}(G^{n+1}).
\end{aligned}
\tag{4.13}
$$

We stop for a moment to give a physical interpretation of 4.13, the same way it is done in [5]. We obtain that the momentum is given by

$$
P^0 = \frac{1}{4}(\mathcal{A}_1 + \dot{\mathcal{D}}_1) \quad P^a = \frac{1}{4}(\dot{\mathcal{A}}_1^a + \ddot{\mathcal{D}}_1^a + \dot{\mathcal{H}}_1^a),
\tag{4.14}
$$

while the displacement of the center of mass from the world line is given by

$$
Y^a = \frac{\mathcal{A}_1^a + \dot{\mathcal{D}}_1^a - \mathcal{H}_1^a}{\mathcal{A}_1 + \mathcal{D}_1}
\tag{4.15}
$$

By using 4.13 we conclude that $\dot{\mathcal{B}}_n^a + \ddot{\mathcal{H}}_n^a$ is related to the change of 3-momentum of the source with respect to the world line, while $\dot{\mathcal{A}}_n^a + \ddot{\mathcal{D}}_n^b$ is related to the change of the displacement of the center of mass, and consequently to the change of the dipole moment of the source. This also implies that $\mathcal{R}^a$ is analogous to the force, while $\mathcal{P}^a$ has no classical analogue. In the same way, without giving the details, the first equation is related to the change of mass monopole, and the last one to the change of the spin angular momentum. While the latter are not a cause of concern, some problems arise regarding the physical interpretation of the second and third equations. Indeed the variation of the dipole moment and the linear momentum reflects the fact that the source is moving with respect to the predetermined world line, which was given ahead of time as a result of the approximation at the previous order. Indeed the variation in time of these quantities is computed with respect to the world line and not to the center of mass of the source. This however implies that we have to take into account an increasingly large number of multipoles in order to describe the motion of the source adequately. In order to avoid this situation, we have to go back before improving the field equations and make sure the world line is chosen such that

$$
\mathcal{R}_n^a - \dot{\mathcal{P}}_n^a = \mathcal{O}(G^{n+1})
\tag{4.16}
$$

In this way we obtain the condition that $\ddot{\mathcal{H}}_n^b - \ddot{\mathcal{A}}_n^b - \dddot{\mathcal{D}}_n^b = \mathcal{O}(G^{n+1})$ and so the second and third equations of 4.13 are solved when all this multipoles are zero. In this way we do not have to deal with a large amount of multipoles anymore. In the next part we will focus on the condition 4.16 and understand how it emerges naturally.

Equations of motion

As concluded at the end of the previous section, before we compute the next approximation of the metric through the iteration of the field equation, it is necessary to adjust the world line. This adjustment can be achieved through a so called retarded Poincaré transformation, which can be seen as a generalisation of a Lorentz transformation. This section has three main goals: the first one is to introduce the retarded Poicaré transformation and describe its properties; the second is to understand how the transformation can be used to pull the field along a new world line so as to maintain the same order of approximation to the field equation; the third is to see how to choose the transformation in a way that the equations of motion are satisfied at every order.

Let us start from the first point. The retarded Poincaré transformation is a diffeomorphism from the causal future of a world line z^a onto the causal future of another world line $z'^{a'}$ in Minkowski spacetime. It is more general than a Lorentz transformation and possesses some useful properties, which we are going to explore.
To begin with, the transformation is defined by

$$y'_a := \Lambda^{a'}{}_b(s_x)x^b + \xi^{a'}(s_x). \tag{5.1}$$

We fix the choice of $\xi^{a'}$ up to an additional constant by imposing

$$\dot{\xi}^{a'} = -\dot{\Lambda}^{a'}{}_b z^b(s). \tag{5.2}$$

We define also the tensor $\eta'_{a'b'}$ and its inverse $\eta'^{a'b'}$, with which we lower and raise primed indices, as

$$\eta'_{a'b'} := \Lambda^c{}_{a'}\Lambda^d{}_{b'}\eta_{cd}, \qquad \eta'^{a'b'} := \Lambda^{a'}{}_c\Lambda^{b'}{}_d\eta^{cd}. \tag{5.3}$$

Using the definition of the retarded Poincaré transformation 5.1, together with 5.2 and 2.2 it follows that

$$\begin{aligned}\frac{\partial y^{a'}}{\partial x^b} &= \frac{\partial}{\partial x^b}\left[\Lambda^{a'}{}_b(s_x)x^b + \xi^{a'}(s_x)\right]\\ &= \Lambda^{a'}{}_b(s_x) - k_b\dot{\Lambda}^{a'}{}_c(s_x)x^c - k_b\dot{\xi}^{a'}\\ &= \Lambda^{a'}{}_b(s_x) - \dot{\Lambda}^{a'}{}_c(s_x)k_b\left(x^c - z^c(s)\right)\\ &= \Lambda^{a'}{}_b(s_x) - r\dot{\Lambda}^{a'}{}_c(s_x)k^c k_b.\end{aligned} \tag{5.4}$$

Because y^a is on the null cone, its s coordinate shares the same value of z^a and we can define a radial coordinate in the primed reference system with the following property:

$$
\begin{aligned}
r' &:= -v'_{a'}(y^{a'} - z'^{a'}) \\
&= -\eta'_{a'b'}\Lambda^{b'}{}_{c}v^{c}(\Lambda^{a'}{}_{c}(s)\,x^{b} + \eta^{a'}(s) - \Lambda^{a'}{}_{b}(s)\,z^{b}(s) - \eta^{a'}(s)) \\
&= -\eta_{cb}\Lambda^{b}{}_{a'}\Lambda^{a'}{}_{b}(x^{b} - z^{b})v^{c} \\
&= -v_{b}(x^{b} - z^{b}) = r.
\end{aligned}
\tag{5.5}
$$

We used that $\eta'_{a'b'}\Lambda^{b'}{}_{c} = \eta_{cb}\Lambda^{b}_{a'}$, which is an immediate consequence of the definitions 5.3. Moreover we can show an analogous property for k^b:

$$
k_b = -\frac{\partial s}{\partial x^b} = -\frac{\partial y^a}{\partial x^b}\frac{\partial s}{\partial y^a} = \Big(\Lambda^{a'}{}_{b}(s_x) - r\dot{\Lambda}^{a'}{}_{c}(s_x)k^c\,k_b\Big)k'_{a'}. \tag{5.6}
$$

Contracting this expression with k^b yields $0 = k_b k^b = \Lambda^{a'}{}_{b}k^b k'_{b'} - r\dot{\Lambda}^{a'}{}_{c}k^c\,k_b k^b\,k'_{b'}$ and so the second term on the right hand side vanishes, leaving $\Lambda^{a'}{}_{b}k^b k'_{a'} = 0$. Instead, contraction with v^a and an easy rearrangement yields

$$
\dot{\Lambda}^{b'}{}_{a}k^a k'_{b'} = -(1 + \Lambda^{b'}{}_{c}v^c k'_{b'})/r = -(1 + v'^{b}k'_{b'})/r = 0, \tag{5.7}
$$

and lets us drop the second term on the right hand side of 5.6, which simplifies to

$$
k_a = \Lambda^{b'}{}_{a}k'_{b'}. \tag{5.8}
$$

Incidentally, 5.7 is also necessary to find an inverse to the transformation of the tensor components. Starting from 5.4, we come up with the ansatz

$$
\frac{\partial x^b}{\partial y^{a'}} = \Lambda^{b}{}_{a'}(s_y) - r\dot{\Lambda}^{b}{}_{d'}(s_y)k^{d'}\,k_{a'}. \tag{5.9}
$$

It is easy to see that when multipling 5.4 and 5.9, the cross products and the product of the second terms vanish thanks to 5.7 and the product of the first terms gives trivially the identity. By applying the transformation 5.1 to a world line z^a and taking the derivative with respect to s of the new world line $z'^{a'}$ we obtain $v'^{a'} = \Lambda^{a'}{}_{b}v^b$ which together with 5.3 implies

$$
\eta'_{a'b'}v'^{a'}v'^{b'} = -1. \tag{5.10}
$$

This means that s is the proper time even for the transformed world line, and we can simplify the notation by dropping the subscripts of s in 5.9.
Summarising, we showed that the retarded Poincaré transformation leaves unchanged the scalar fields s and r and the null vector k^a, which means in turn that the future null cone is preserved. For any world line there are many retarded Poincaré transformations which differ from one another by a spatial rotation. To address this problem let us consider a tetrad basis attached to the world line z^a. The general way to transport this basis can be written in the following way:

$$
\dot{e}_i^{a'} = -\Omega'^{a'}_{b'}e_i^{b'}, \tag{5.11}
$$

where $\Omega'^{a'}_{b'}$ is given by

$$
\Omega'^{a'}_{b'} = \dot{v}'^{a'}v'_{b'} - v'^{a'}\dot{v}'_{b'} + v'_{c'}\,\omega'_{d'}\,\epsilon^{a'}{}_{b'}{}^{c'd'}. \tag{5.12}
$$

The first two terms represent the contribution from the Fermi-Walker transport, which leaves the spacelike tetrad fields unchanged, while the third term, proportional to the angular velocity tensor $\omega'_{a'}$ and antisymmetric, acts as a spatial rotation and leaves the timelike tetrad field constant. We neglect this last term. With this choice we pick up a single retarded Poincaré transformation among the infinite ones that transport the world line $z^a = (s, 0, 0, 0)$ to $z'^{a'}(s)$ and include a rotation of the coordinate basis.
As we know, the transformations that preserve the Minkowski metric are the Lorentz transformations and therefore we consider changes of basis of the form $e^{a'} = \Lambda^{a'}{}_b e^b$, in which the matrices $\Lambda^{a'}{}_b$ depend on the spacetime coordinates [7]. By plugging the transformation law into 5.11 we obtain

$$\dot{\Lambda}^{a'}{}_b = -\Omega'^{a'}{}_{c'}\Lambda^{c'}{}_b, \tag{5.13}$$

which has a unique solution given suitable initial conditions.

Now that we have introduced the essential tools, we see how to implement them in order to achieve the transformation compatibly with our expansion scheme. Let us consider a single source and split the field at every order into a background h_B^{ab} and a self-field part h_A^{ab},

$$h_n^{ab} = h_A^{ab} + h_B^{ab}. \tag{5.14}$$

The self-field contains at least the $\mathcal{O}(G)$ part of the field produced by the source itself, while the background contains at least the $\mathcal{O}(G)$ part of all the other sources. Let us define the transformation with $\Lambda^{a'}{}_b(s=0) = \delta_b^{a'}$ to match smoothly to the initial data and with $\dot{\Lambda}(s) = \mathcal{O}(G^n)$. In this transformation the background field is the same function in the new coordinates as it was in the old ones, while the self-field is boosted and accelerated through a time dependent Lorentz transformation

$$\begin{aligned} h'^{a'b'}_A(y) &= \Lambda^{a'}{}_c\Lambda^{b'}{}_d h_A^{cd}(x) + \delta_c^{a'}\delta_d^{b'}h_B^{cd}(y) \\ &= \delta_c^{a'}\delta_d^{b'}h_n^{cd}(y) + \Big(\Lambda^{a'}{}_c\Lambda^{b'}{}_d h_A^{cd}(x) - \delta_c^{a'}\delta_d^{b'}h_A^{cd}(y)\Big) \end{aligned} \tag{5.15}$$

In the second line we just rewrote the expression in a way that does not refer explicitly to the background field. In particular, the expression in brackets vanishes for $s = 0$ and is proportional to h_A^{ab}, so that we can approximate it with $s\dot{\Lambda}h_A = \mathcal{O}(sG^{n+1})$. To verify that the accuracy of the approximation is maintained we consider $E^{a'b'}(h'^{a'b'}(y))$ and split it into its linear and quadratic part

$$\begin{aligned} E^{a'b'}(h'^{a'b'}(y)) = E^{a'b'}(\delta_c^{a'}\delta_d^{b'}h_n^{cd}(y)) + E_{lin}^{a'b'}(\Lambda^{a'}{}_c\Lambda^{b'}{}_d h_A^{cd}(x) - \delta_c^{a'}\delta_d^{b'}h_A^{cd}(y)) + \\ + E_\tau^{a'b'} + \mathcal{O}(s^2G^{2n+2}). \end{aligned} \tag{5.16}$$

Here E_τ depends both on $\Lambda^{a'}{}_c\Lambda^{b'}{}_d h_A^{cd}(x) - \delta_c^{a'}\delta_d^{b'}h_A^{cd}(y)$ and $\delta_c^{a'}\delta_d^{b'}h_n^{cd}(y)$, while terms which are quadratic in $\Lambda^{a'}{}_c\Lambda^{b'}{}_d h_A^{cd}(x) - \delta_c^{a'}\delta_d^{b'}h_A^{cd}(y)$ are of higher order. With the observations carried out so far we can conlcude that

$$E^{a'b'}(h'^{a'b'}(y)) = \mathcal{O}(G^{n+1}) + \mathcal{O}(sG^{n+2}). \tag{5.17}$$

This means that the approximation is preserved as long as $s = \mathcal{O}(G^{-1})$, which is in general satisfied.

At this point we can see how the transformation 5.15 allows for the equations of motion to be satisfied. In what follows we will assume for simplicity that at the first order $\mathcal{A}_1$ is the the only $\mathcal{O}(G)$ mode, while the others are $\mathcal{O}(G^2)$ and negligible. Looking back at 3.7 we will consider the terms linear in $\mathcal{A}_1$, so that we can pull the latter out of the derivative and write

$$M_1^{abL} = \mathcal{A}_1^L \partial_L \left(\frac{1}{r} v^a v^b \right). \tag{5.18}$$

We will refer to the components of $\mathcal{P}_n^a$ and $\mathcal{R}_n^a$ which are linear in $\mathcal{A}_1$ and therefore contain $\dot{v}^a$ or $\ddot{v}^a$ respectively as $\mathcal{P}_\mathcal{A}^a$ and $\mathcal{R}_\mathcal{A}^a$. These tensors are $\mathcal{O}(G^2)$. To obtain an expression for them, first we need to find the part of the Einstein's tensor density which is linear in $\mathcal{A}_1$,

$$E_A^{ab} = -\frac{2}{r}\mathcal{A}_1 \left[\ddot{v}^{(a}k^{b)} + \frac{1}{r}\dot{v}^{(a}(k^{b)} - v^{b)}) + \dot{v}^{(a}k^{b)}k_c\dot{v}^c - \frac{1}{2}\eta^{ab}\left(k_c\ddot{v}^c + \frac{1}{r}k_c\dot{v}^c + (k_c\dot{v}^c)^2 \right) \right]. \tag{5.19}$$

We skipped the details of this tedious calculation which require only basic relations, namely 2.9 and 2.5. Also, we can neglect all the combinations of the velocity and its derivatives which are $\mathcal{O}(G^2)$ as $\mathcal{A}_1$ is already $\mathcal{O}(G)$. Now we can extract from 3.7 the monopole and dipole parts and relate them to $\mathcal{P}_\mathcal{A}^a$ and $\mathcal{R}_\mathcal{A}^a$. In particular for the monopole part we obtain

$$\begin{aligned} \partial_a p_{n\ (l=0)}^{ab} &= -\sum_{m=0}^{\infty} b_{2m,m} \int ds \frac{r_0^{2m}}{(2m)!}\frac{d^{2m}}{ds'^{2m}}\left[\frac{1}{4\pi}\oint_{r_0,s} r_0^2 \partial_a r E_\mathcal{A}^{ab}(s'-r_0)\sin\theta\, d\theta\, d\phi \right] \\ &= -\sum_{m=0}^{\infty} \frac{1}{2m+1}\int ds \frac{r_0^{2m}}{(2m)!}\frac{d^{2m}}{ds^{2m}}\left[\mathcal{A}_1(\dot{v}^b + r_0\ddot{v}^b) + \mathcal{O}(G^3) \right]_{s-r_0}. \end{aligned} \tag{5.20}$$

The symbol $\mathcal{O}(G^3)$ can be pulled out of the brackets as the prefactors do not carry any dependence on G. Using 4.10 we can compare the two expressions and conclude that

$$K_{\mathcal{A}\ (l=0)}^b(s) = \mathcal{R}_\mathcal{A}^b(s) = -\mathcal{A}_1 \sum_{m=0}^{\infty}\left[\frac{r_0^{2m}}{(2m+1)!}\frac{d^{2m}}{ds^{2m}}(\dot{v}^b + r_0\ddot{v}^b) \right]_{s-r_0} + \mathcal{O}(G^3) \tag{5.21}$$

We can proceed in the same way for the dipole part,

$$\begin{aligned} \partial_a p_{n\ (l=1)}^{ab} &= -\sum_{m=0}^{\infty} b_{2m+1,m}\int ds \frac{r_0^{2m}}{(2m+1)!}\frac{d^{2m}}{ds'^{2m}}\left[\frac{1}{4\pi}\oint_{r_0,s} r_0^3\partial_a r E_\mathcal{A}^{ab}(s'-r_0)n^d \sin\theta\, d\theta\, d\phi \right]\times \\ &\quad \times \partial_d G(x - z(s)) \\ &= -\sum_{m=0}^{\infty}\frac{3}{2m+3}\int ds \frac{r_0^{2m}}{(2m+1)!}\frac{d^{2m}}{ds^{2m}}\left[-\frac{1}{3}\mathcal{A}_1 r_0^2 v^b\ddot{v}^b + \mathcal{O}(G^3) \right]_{s-r_0} \partial_d G(x-z(s)) \\ &= \sum_{m=0}^{\infty}\int ds \frac{r_0^{2m+2}}{4\pi(2m+3)(2m+1)!}\frac{d^{2m}}{ds^{2m}}\left[\mathcal{A}_1 v^b\ddot{v}^b \right]_{s-r_0} \partial_d G(x-z(s)) + \mathcal{O}(G^3) \end{aligned} \tag{5.22}$$

In both cases the solution of the integrals in square brackets are given in [1], as well as the value of the coefficients. The general way to obtain them is identical to how we obtained 2.30.

The sign difference between 5.20 and 5.22 is due to an alternating sign in 4.9. By comparison with 4.10, this time we take only the part which is linear in v^b,

$$K^b_{\mathcal{A}\ (l=1)}(s) = \mathcal{P}^b_{\mathcal{A}}(s) = \mathcal{A}_1 \sum_{m=0}^{\infty} \left[\frac{r_0^{2m+2}}{(2m+3)(2m+1)!} \frac{d^{2m}}{ds^{2m}} \ddot{v}^b \right]_{s-r_0} + \mathcal{O}(G^3). \tag{5.23}$$

At this point we can put the two expressions together

$$\begin{aligned}
[\mathcal{R}^b_A - \dot{\mathcal{P}}^b_A](s) &= -\mathcal{A}_1 \dot{v}^b \Big|_{s-r_0} - \mathcal{A}_1 \sum_{m=0}^{\infty} \left[\frac{r_0^{2m+2}}{(2m+3)!} \frac{d^{2m}}{ds^{2m}} \dddot{v}^b + \frac{r_0^{2m+1}}{(2m+1)!} \frac{d^{2m}}{ds^{2m}} \ddot{v}^b + \right. \\
&\quad \left. + \frac{r_0^{2m+2}}{(2m+3)(2m+1)!} \frac{d^{2m}}{ds^{2m}} \dddot{v}^b \right]_{s-r_0} + \mathcal{O}(G^3) \\
&= -\mathcal{A}_1 \dot{v}^b \Big|_{s-r_0} - A_1 \sum_{m=0}^{\infty} \left[\frac{r_0^{2m+2}}{(2m+2)!} \frac{d^{2m}}{ds^{2m}} \dddot{v}^b + \frac{r_0^{2m+1}}{(2m+1)!} \frac{d^{2m}}{ds^{2m}} \ddot{v}^b \right]_{s-r_0} + \mathcal{O}(G^3) \\
&= -\mathcal{A}_1 \left[\dot{v}^b + \sum_{m=0}^{\infty} \left[\frac{r_0^{2m+2}}{(2m+2)!} \frac{d^{2m+2}}{ds^{2m+2}} \dot{v}^b \right] + \sum_{m=0}^{\infty} \left[\frac{r_0^{2m+1}}{(2m+1)!} \frac{d^{2m+1}}{ds^{2m+1}} \dot{v}^b \right] \right]_{s-r_0} + \mathcal{O}(G^3) \\
&= -\mathcal{A}_1 \sum_{m=0}^{\infty} \left[\frac{r_0^m}{m!} \frac{d^m}{ds^m} \dot{v}^b \right]_{s-r_0} + \mathcal{O}(G^3) \\
&= -\mathcal{A}_1 \dot{v}^b + \mathcal{O}(G^3).
\end{aligned} \tag{5.24}$$

To go from the third to the fourth line we realised that the first two terms in square brackets contribute to the even terms of a series where we replace $2m+m$ with m, while the third term in square bracket contributes to the odd terms in the same series. We realise at this point that this expression is exactly the Taylor series of $-\mathcal{A}_1 \dot{v}^b$ around $s - r_0$, but evaluated at s, like the left hand side of the equation. Let us consider now a retarded Poincaré transformation performed before the n-th iteration of the field equations. The acceleration of the world line will be modified according to

$$\dot{v}_n^{b'} = \Lambda^{b'}{}_a \dot{v}^a_{n-1} + \dot{\lambda}^{b'}{}_a v^a_{n-1}. \tag{5.25}$$

Assume that $\Lambda^{b'}{}_a(s)$ is determined everywhere except for some value s_0, let us define $F^{b'} := [\mathcal{R}^b_A - \dot{\mathcal{P}}^b_A](s_0)$ and temporarily $\dot{\Lambda}^{b'}{}_a(s_0) = 0$. Now, for $\dot{\Lambda}^{b'}{}_a(s_0) = \mathcal{O}(G^n)$ different from zero, we can enforce the equations of motion at s_0 too by imposing that

$$\mathcal{A}_1 \dot{\Lambda}^{b'}{}_a(s_0) v^a_{n-1} = F^{b'} \tag{5.26}$$

Indeed, using 6.9 and 5.25, we obtain

$$\mathcal{A}_1 \Lambda^{b'}{}_a \dot{v}^a_{n-1} = \mathcal{O}(G^{n+1}). \tag{5.27}$$

This represents three differential equations for the six free parameters of $\Lambda^{b'}{}_a$. The remaining three are fixed by 5.13. Using 6.9 we can rewrite this last relation in the form of an iteration of the equations of motion for the source, with $F^{b'}$ representing a residual force remaining on the world line $z_{n-1}(s)$:

$$\mathcal{A}_1 \dot{v}_n^{b'} = \mathcal{A}_1 \Lambda^{b'}{}_a \dot{v}^a_{n-1} + F^{b'}. \tag{5.28}$$

Behaviour at future null infinity

We turn our attention to the asymptotic behaviour of the radiation field. In particular we will study the limit at future-null infinity, obtained by computing the limit of large r while keeping $s = t - r$ constant. We will see that some problems arise when trying to carry out a multipole expansion in this limit. The aim of this section is to give an idea of the main steps needed to extend the iterative procedure explained previously to this asymptotic regime. We consider an isolated system and keep using outgoing-null spherical coordinates, but attached to a nonaccelerating world line near the center of the system. We will verify that in this limit the metric admits at every order an expansion in inverse powers of r, along with a simultaneous expansion in G. In particular h_1^{ab} will admit a multipolar decomposition like in 3.7, with the $\mathcal{O}(G^2)$ part also being $\mathcal{O}(r^{-2})$,

$$h_1^{ab} = \frac{1}{r}\chi_1^{ab}(s,\theta,\phi) + \mathcal{O}(Gr^{-2}). \tag{6.1}$$

It is clear then that χ^{ab} represents the dominant part of the outgoing radiation at large distances. However, a problem emerges already at second order when trying to evaluate p_2^{ab}. As it is shown in [6] we can write for an arbitrary integer M

$$p_2^{ab} = N^L \sum_{k=1}^{M} \frac{1}{r^k}\left(T_k^L(s) + log(r)U_k^L(s)\right) + V_N^L(r,s), \tag{6.2}$$

where the remainder V_N^L is $\mathcal{O}(r^{-N})$. The problem resides in the logarithms as they make the post-Minkowski expansion ill defined: the n-th approximation becomes larger than the $(n+1)$-th at sufficiently large distances. The presence of this logarithmic terms can be seen as a consequence of the surfaces of constant null coordinate s in the region described by outgoing-null coordinates that do not match the future null cones in the wave region (cf. [8]). This mismatch can be solved by a gauge transformation, which we are going to describe next.
As a consequence of the restriction on the form of ntioned at the end of that section, the form of the field χ^{ab} is restrained as well:

$$k_a\chi_1^{ab} = -v^b(\mathcal{A}_1 + \dot{\mathcal{D}}_1) - (\mathcal{B}_1^b + \dot{\mathcal{H}}_1^b). \tag{6.3}$$

We note that if the right hand side of this equation were to vanish, then the linear part of 2.9 would vanish and the dominant contribution to the Einstein tensor density would come from τ_{ab}. More precisely it would be equal to

$$E_{n-1}^{ab} = -\frac{1}{r^2}k^a k^b \Psi_n(s,\theta,\phi) + \bar{E}_{n-1}^{ab}, \tag{6.4}$$

where $\bar{E}_{n-1}^{ab} = \mathcal{O}(G^2 r^{-3})$ and $\Psi_n(s,\theta,\phi)$ contains contractions of derivatives of χ_{ab}. But we can always apply a gauge transformation to set 6.3 equal to zero. Let us remember the expression for the momentum in terms of multipoles 4.14 and conclude that the second term in 6.3 can be made to vanish after a boost. To get $\mathcal{D}=0$ we use another transformation with $\Box\lambda^a=0$, described in [5] (eq. 2.9b). This condition is the harmonic gauge, in which the field equations 2.16 are satisfied with $H^b(n)=0$ and the general form of the metric is preserved. Finally, with the condition $\lambda_0=\frac{1}{2}\mathcal{A}_1\log r$, $\lambda^i=-\frac{1}{4}\mathcal{A}_1 n^i$, we obtain at leading order in $\frac{1}{r}$ and neglecting the terms in $\dot{\mathcal{A}}_1$,

$$h_1^{ab}=\frac{1}{2r}\mathcal{A}_1 k^a k^b, \tag{6.5}$$

where we used 2.5. We can easily see that indeed now $k_a\chi^{ab}=0$.
At the generic order n we assume that the metric has an expansion in inverse powers of r, in analogy with what was done before only in G before,

$$\begin{aligned} h_{n-1}^{ab} &= \frac{1}{r}\chi_{n-1}^{ab}(s,\theta,\phi)+\mathcal{O}(Gr^{-2}) \\ \delta\chi_{n-1}^{ab} &= \chi_{n-1}^{ab}-\chi_{n-2}^{ab}, \end{aligned} \tag{6.6}$$

under the condition that

$$k_a\chi_{n-1}^{ab}=0. \tag{6.7}$$

In the same way we have just proceeded at first order, we have

$$E_{n-1}^{ab}=-\frac{1}{r^2}k^a k^b\Psi_n(s,\theta,\phi)+\bar{E}_{n-1}^{ab} \tag{6.8}$$

and we look for an improved solution $h_n^{ab}=h_{n-1}^{ab}+\delta h_n^{ab}$. This can be plugged into 2.9 to obtain

$$E^{ab}(h_{n-1}+\delta h_n)=\mathcal{O}(G^{n+1}r^{-3}). \tag{6.9}$$

We can go through the same passages described in the previous section to construct the general solution at the n-th order, but we encounter an obstacle. When we evaluate the third line of 4.7 we cannot anymore neglect the second integral, because in the wave zone we have 6.9. Instead we end up with the modified condition

$$\partial_a p_n^{ab}+\partial_a q_n^{ab}=\frac{1}{r^2}\zeta_{n+1}^b(s,\theta,\phi). \tag{6.10}$$

We can define a tensor γ^{ab} such that its derivative is equal to the right hand side of 6.10:

$$\partial_a\gamma^{ab}=-k_a\dot{\gamma}_{n+1}^b=-\frac{1}{r^2}\zeta_{n+1}^b+\mathcal{O}(r^{-3}), \tag{6.11}$$

which leads us to an expression of the same form of 4.4, as desired,

$$\partial_a p_n^{ab}+\partial_a q_n^{ab}+\partial_a\gamma_{n+1}^{ab}=\mathcal{O}(G^{n+1}r^{-3}) \tag{6.12}$$

Moreover, by taking the second derivative of γ^{ab} we obtain

$$\Box\gamma_{n+1}^{ab}=k_c k^c\ddot{\gamma}_{n+1}^{ab}+\mathcal{O}(G^{n+1}r^{-3})=\mathcal{O}(G^{n+1}r^{-3}). \tag{6.13}$$

This is fundamental as it implies that the metric obeys the same field equations even after the introduction of the correcting term γ^{ab}

$$\Box(h_n^{ab} + \gamma_{n+1}^{ab}) = E_{n-1}^{ab} + \mathcal{O}(G^{n+1}r^{-3}). \tag{6.14}$$

Eventually, a further correction $\partial\lambda_n^{ab} = \mathcal{O}(G^n)$ that leaves the linear part of the operator E^{ab} invariant is required to eliminate the logarithmic terms. However we will not investigate the details of such a transformation.
Summing up, the new iterative step to construct the metric at the next order is given by

$$\delta h_n^{ab} := p_n^{ab} + q_n^{ab} + \delta\lambda_n^{ab} + \gamma_{n+1}^{ab}. \tag{6.15}$$

With this and 2.19 we have obtained a proper power expansion in inverse powers of r with the ansatz 6.6 holding at every order.
To conclude the iterative procedure we should prove 6.7 and 6.8. For the former, note that, since $\delta h_n^{ab} = \frac{1}{r}\delta\chi_n^{ab}(s,\theta,\phi) + \mathcal{O}(G^n r^{-2})$, its derivative is equal to

$$\partial_a \delta h_n^{ab} = -\frac{1}{r}k^a \delta\dot{\chi}_n^{ab} + \mathcal{O}(G^n r^{-2}) \tag{6.16}$$

But from 6.11 it must be that $\partial_a \delta h_n^{ab} = \mathcal{O}(G^n r^{-2})$, so $k_a \delta\dot{\chi}_n^{ab} = 0$. Because $\delta\chi_n^{ab} = 0$ on the initial hypersurface for $n > 1$, $k_a \delta\chi_n^{ab} = 0$ everywhere. So 6.7 is proven by induction using 6.6b. We have neglected here the analysis of the term $\delta\lambda_n^{ab}$, but for a rigorous analysis we should make sure that its derivative does not produce terms of lower order. To check 6.8, we have to go back to consider 2.22 and plug the new definition 6.15 for the iterative step. Using 6.14 and the property of the transformation $\delta\lambda_n^a$ stated above we obtain

$$\begin{aligned} E^{ab}(h_n) = {} & \partial^a\partial_c(p_n^{cb} + q_n^{cb} + \gamma_{n-1}^{cb}) + \partial^b\partial_c(p_n^{ca} + q_n^{ca} + \gamma_{n-1}^{ca}) - \eta^{ab}\partial_c\partial_d(p_n^{cd} + q_n^{cd} + \gamma_{n-1}^{cd}) + \\ & - \tau^{ab}(h_n) + \tau^{ab}(h_{n-1}) + \mathcal{O}(G^{n+1}r^{-3}). \end{aligned} \tag{6.17}$$

Now it can be shown that indeed 6.8 holds at every order with a recursive expression for $\Psi_n(s,\theta,\phi)$.

Conclusions

We presented a method to improve the accuracy of an approximate solution to Einstein's equations which can be iterated to any order. Let us quickly recap the crucial passages: the first step was to construct a linear solution h_1^{ab}; secondly $\mathcal{P}^b$ and $\mathcal{R}^b$ were computed, along with the boundary conditions for E^{ab}; the world line was then adjusted by means of a retarded Poincaré transformation, which preserved the accuracy of the approximation, and the field equations were iterated by computing q^{ab} and p^{ab} at the next order; eventually a small correction γ^{ab} and a transformation $\delta\lambda^{ab}$ were added to the metric to enforce a proper behaviour at future null infinity.
A weakness of this approach is the difficulty to relate the physics of the sources to the conditions at the inner boundaries in the presence of tidal forces. Instead, a strength of this approach is that it can be possibly implemented in a straightforward way to be run by a computer. However, there also exist analytical approaches to compute post-Minkowski approximations at more than linear order. One example is given by [9], where techniques from perturbative field theory are used to compute relevant physical quantities up to third post-Minkowski order for a system of binary spinless objects and compared with state-of-the-art results for post-Newtonian expansions.

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