Master Thesis

# The Boundary Terms of the Einstein-Hilbert Action

## and alternative spacetime geometries of General Relativity


Submitted by

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*To Aurélie*

# Abstract

We study the variational problem of the Einstein-Hilbert action for spacetime manifolds with boundaries. In order to make the problem well-posed, the Einstein-Hilbert action must be supplied with a counter-term. We derive the counter-terms for non-null boundaries, known as the Gibbons-Hawking-York boundary term, and for null-like boundaries, recently discovered by K. Parattu *et al.* [8]. Then, using the tetrad formulation of the Einstein-Hilbert action, we show that both boundary terms can be derived with less effort. Furthermore, we study the teleparallel and symmetric teleparallel formulations of general relativity which have the advantage of already incorporate the boundary term. We compare these three equivalent descriptions of general relativity, which differ only in the boundary term, calculating the Euclidean action of a Schwarzschild black hole. At last, we use this result to compute the entropy of a Schwarzschild black hole as S. Hawking first did in 1977 [9].

# Contents

# 1 Introduction

The starting point of any field theory is the action principle. The action for general relativity was first introduced by D. Hilbert in 1915 [1] and it is known as the Einstein-Hilbert (EH) action

$$S_{EH} = \frac{c^4}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} R \,. \tag{1.1}$$

The variation of this functional with respect to the dynamical variable, i.e. the metric $g_{\mu\nu}$, is responsible to deliver the Einstein field equations in vacuum.

In contrast to classical field theories, e.g. electromagnetism, the Einstein-Hilbert action contains second order derivatives of the dynamical variable. The second order derivatives of the metric make their appearance in the boundary term when varying the action. Hilbert, in his original paper, argued that by fixing the metric at infinity the boundary term vanishes and it does not contribute to the variation of the action [2] [3]. However, when we are dealing with spacetime manifolds with boundaries, this term cannot be ignored, otherwise one will implicitly assume boundary conditions which will overdetermine the theory [4].

The standard solution to this problem is to correct the Einstein-Hilbert action by adding a counter-term which fixes a variable on the boundary. This term is by no mean unique but it can be defined in ways which are more convenient than others. The fact that the variational principle of general relativity is not a well-posed problem was already brought up by Einstein [5]. In his paper, Einstein proposes to correct the EH action with a counter-term which is not covariant [6] and therefore does not solve the problem.

The most-popular covariant counter-term to the EH action is the Gibbons-Hawking-York boundary term [7], explicitly derived in section 2.2. The covariance is achieved by introducing a new variable normal to the boundary surface in addition to the metric. The GHY counter-term is applicable only to non-null boundaries. In section 2.3 we derive a counter-term for null boundaries which was recently discovered by K. Parattu *et al.* [8]. Then, in section 2.4, we introduce the tetrad formalism and use the tetrad description of general relativity to derive both boundary terms.

A different approach to this problem requires the introduction of alternative spacetime geometries of general relativity. In sections 3.3 and 3.4 we show that the teleparallel action of Weitzenböck spacetime and the symmetric teleparallel action of symmetric teleparallel spacetime, which differ from the conventional general relativity by a divergence term, already incorporate the boundary term. Teleparallel general relativity achieves this result describing the effects of gravity through the torsion and not by the curvature of spacetime, while in symmetric telerallel spacetime both the torsion and curvature of spacetime vanish and the effects of gravity are described by the non-metricity tensor.

At last, in section 4 we compare these different formulations of general relativity and their boundary terms computing the Euclidean action of a Schwarzschild black hole. We then use this result to calculate the entropy of the black hole as first done by Stephen Hawking [9].

The convention we are going to use, if not stated otherwise, for the Riemann tensor is

$$R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\lambda\beta}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\mu\beta} \tag{1.2}$$

where the Greek indices run from 0 to 3. Furthermore, the lower case Latin indices run from 1 to 3, the metric has signature $(-,+,+,+)$ and from now on we will set $c = G = \hbar = k_B = 1$.

# 2 The boundary terms of the Einstein-Hilbert action

## 2.1 Overview

In this section we study the variational problem of the Einstein-Hilbert action for spacetime manifolds with boundaries. First of all, in section 2.2, we explicitly derive the Gibbons-Hawking-York counter-term through the variation of the EH action. Second, in section 2.3, we derive the counter-term to the EH action for null-like boundaries. At last, in section 2.4, we use the tetrad formalism to show that both boundary terms can be derived with less effort from the tetrad Einstein-Hilbert action.

## 2.2 Gibbons-Hawking-York counter-term

### 2.2.1 Introduction

Standard textbooks do not provide an explicit derivation of the Gibbons-Hawking-York (GHY) boundary term, see for example E. Poisson [10], T. Padmanabhan [11] and S. Hawking [7] where they directly list the result and show that its variation compensates the boundary term in the variation of the Einstein-Hilbert action. This makes the result somewhat mysterious because it is not obvious and one should also know a priori that the induced metric $h_{\alpha\beta}$ has to be held fix on the boundary rather than the full metric $g_{\mu\nu}$.

After a short overview of the mathematics, we explicitly derive the GHY boundary term following the steps presented in T. Padmanabhan's paper [12]. Our aim is to bring the variation of the Einstein-Hilbert action in the following form

$$\delta S_{EH} = \int_{\mathcal{M}} d^4x\,(\text{Equation of Motion Term})\,\delta(\text{Dynamical Variable}) \tag{2.1}$$

$$+ \int_{\partial\mathcal{M}} d^3x\,(\text{Conjugate Momentum})\,\delta(\text{Variables to be fixed}) \tag{2.2}$$

$$+ \int_{\partial\mathcal{M}} d^3x\,\delta(\text{Boundary Term}) + \int_{\partial\mathcal{M}} d^3x\,(\text{Total Divergence Term})\,, \tag{2.3}$$

from which we will be able to read out the new variable to held fix on the boundary along with the dynamical variable. Note that doing so, we have to pay attention to the degrees of freedom of the theory and be sure we are not overdetermining the theory. Finally, we will neglect the divergence term and recover the boundary term.

### 2.2.2 Mathematical framework

Here we will briefly review the mathematics of non-null hypersurfaces in four-dimensional spacetime that we will need in the derivation of the GHY boundary term. The contents presented here can be found in the textbooks of E. Poisson [10] and E. Gourgoulhon [13], here we will stick to the convention of E. Poisson's book.

A *hypersurface* is a three-dimensional submanifold embedded in four-dimensional spacetime manifold. This surface can be either spacelike, timeline or null. The hypersurface $\Sigma$ is described by a scalar function which satisfy

$$\Phi(x^\alpha) = 0\,, \tag{2.4}$$

or by a parametric equation

$$x^\alpha = x^\alpha(y^a)\,, \tag{2.5}$$

where $y^a$, with $a = 1, 2, 3$, are coordinates intrinsic to the hypersurface.

Once the restriction $\Phi$ on the coordinates is defined and if the hypersurface is non-null, a *unit normal* vector $n_\alpha$ to $\Sigma$ can can introduced. We define $n_\alpha$ so that $n^\alpha$ point in the direction of increasing $\Phi$, i.e. $n^\alpha \Phi_{,\alpha} > 0$, as follows

$$n_\alpha = \frac{\varepsilon \Phi_{,\alpha}}{|g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}|^{\frac{1}{2}}}\,, \tag{2.6}$$

where we defined $\varepsilon$ to be the square of the hypersurface unit normal, i.e.

$$\varepsilon = n_\mu n^\mu = \begin{cases} +1 \text{ timelike} \\ -1 \text{ spacelike} \end{cases}\,, \tag{2.7}$$

in order to generalize the result to both spacelike and timelike surfaces. Note that $n_\alpha$ cannot be defined for a null surface because $g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$ would be equal to zero.

The three-metric induced on the hypersuface $\Sigma$ is obtained by restricting the line element to displacements confined to the hypersurface as follows

$$\begin{aligned} ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b}dy^b\right) = \\ = g_{\alpha\beta}e^\alpha_a e^\beta_b dy^a dy^b \equiv h_{ab}dy^a dy^b\,, \end{aligned} \tag{2.8}$$

where $e^\alpha_a := \frac{\partial x^\alpha}{\partial y^a}$ are vectors tangent to curves contained in $\Sigma$, i.e. it holds $n_\alpha e^\alpha_a = 0$, and $h_{ab} := g_{\alpha\beta}e^\alpha_a e^\beta_b$ is the *induced metric* of the hypersurface. Note that $h_{ab}$ transforms as a scalar under spacetime coordinate transformations $x^\alpha \to x'^\alpha$ and as a tensor under hypersurface coordinates transformations $y^a \to y^b$. Furthermore, given the induced metric and the normal to the hypersurface one can recover the metric according to

$$g_{\alpha\beta} = h_{ab}e^a_\alpha e^b_\beta + \varepsilon n_\alpha n_\beta \equiv h_{\alpha\beta} + \varepsilon n_\alpha n_\beta\,. \tag{2.9}$$

This relation is verified by computing all inner products between $n^\alpha$ and $e^\alpha_a$.

Now, using the projector operator

$$h^\alpha{}_\beta = \delta^\alpha{}_\beta - \varepsilon n^\alpha n_\beta \tag{2.10}$$

we define the *induced covariant derivative* on the hypersurface $\partial\mathcal{M}$ as

$$D_\alpha A_\beta := h^\mu{}_\alpha h^\nu{}_\beta \nabla_\mu A_\nu\,, \tag{2.11}$$

$$D_\alpha A^\beta := h^\mu{}_\alpha h^\beta{}_\nu \nabla_\mu A^\nu\,, \tag{2.12}$$

for any covariant tensor $A_\beta$ and contravariant tensor $A^\beta$ respectively. We can derive a useful relation between $\nabla_\mu$ and $D_\mu$ for a vector field $\mathbf{u}$ along another vector field $\mathbf{v}$ when both vectors are tangent to the hypersurface $\Sigma$. We proceed as follows

$$\begin{aligned} (D_\mathbf{v}\mathbf{u})^\alpha = v^\lambda D_\lambda u^\alpha = v^\lambda h^\mu{}_\lambda h^\alpha{}_\nu \nabla_\mu u^\nu = v^\mu(\delta^\alpha{}_\nu - \varepsilon n^\alpha n_\nu)\nabla_\mu u^\nu = \\ = v^\mu \nabla_\mu u^\alpha - \varepsilon v^\mu n^\alpha n_\nu \nabla_\mu u^\nu = v^\mu \nabla_\mu u^\alpha + \varepsilon v^\mu n^\alpha u^\nu \nabla_\mu n_\nu\,, \end{aligned} \tag{2.13}$$

where in the last equality we used $n_\nu u^\nu = 0$ to write $n_\nu \nabla_\mu u^\nu = -u^\nu \nabla_\mu n_\nu$, since $n^\alpha$ and $u^\beta$ are orthogonal. Rearranging the terms of eq. (2.13) we get

$$\nabla_\beta u^\alpha = D_\beta u^\alpha - \varepsilon n^\alpha u^\nu \nabla_\beta n_\nu \,, \tag{2.14}$$

then, if we define $a_\nu := n^\alpha \nabla_\alpha n_\nu$, we get the following expression for the divergence of $u^\alpha$

$$\nabla_\alpha u^\alpha = D_\alpha u^\alpha - \varepsilon a_\nu u^\nu \,. \tag{2.15}$$

We introduce, see Poisson's book for the proof, the *surface element* of a non-null hypersurface

$$d\Sigma = \sqrt{|h|}d^3y \,, \tag{2.16}$$

where $h \equiv \det h_{ab}$. Furthermore, the *directed* surface element that points in the direction of increasing $\Phi$ is $n_\alpha d\Sigma$ and for non-null surfaces we define

$$d\Sigma_\alpha := \varepsilon n_\alpha d\Sigma \,. \tag{2.17}$$

In the next section we will use two important theorem of differential geometry which are the *Gauss' theorem* and the *Stokes' theorem.* The first one states that for any vector field $A^\alpha$ defined within a finite region of spacetime manifold $\mathcal{M}$, bounded by a closed hypersurface $\partial\mathcal{M}$, the following holds

$$\int_{\mathcal{M}} d^4x \sqrt{-g}\, \nabla_\alpha A^\alpha = \int_{\partial\mathcal{M}} d\Sigma_\alpha\, A^\alpha \,. \tag{2.18}$$

While the second one states that for any antisymmetric tensor field $B^{\alpha\beta}$ in a three-dimensional region of the hypersurface $\Sigma$, bounded by a closed two-surface $\partial\Sigma$, the following holds

$$\int_{\Sigma} d\Sigma_\alpha\, \nabla_\beta B^{\alpha\beta} = \frac{1}{2}\int_{\partial\Sigma} dS_{\alpha\beta}\, B^{\alpha\beta} \,. \tag{2.19}$$

### 2.2.3 Derivation of the Gibbons-Hawking-York counter-term

We begin with the following well known result given in every standard textbook of general relativity, see for example the one of E. Poisson [10], to compute the variation of the Einstein-Hilbert action

$$\begin{aligned} 16\pi\, \delta S_{EH} &= \int_{\mathcal{M}} d^4x \sqrt{-g} G_{\mu\nu}\delta g^{\mu\nu} + \int_{\mathcal{M}} d^4x \sqrt{-g}\nabla_\mu V^\mu = \\ &= \int_{\mathcal{M}} d^4x \sqrt{-g} G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^3y\, \varepsilon\sqrt{|h|} n_\mu V^\mu \end{aligned} \tag{2.20}$$

where $V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\beta\alpha}$ and we used Gauss' theorem to get from the first to the second expression. Here $h$ is the determinant of the induced metric on the boundary surface $\partial\mathcal{M}$ and $n_\mu$ is the unit normal to $\partial\mathcal{M}$ which is normalized, as stated in eq. (2.7), as $\varepsilon = n_\mu n^\mu \equiv \pm 1$ corresponding to timelike and spacelike parts of the boundary $\partial\mathcal{M}$.

To simplify the expression in the second term of eq. (2.20), we use the following relations

$$\delta(\nabla_\mu n_\nu) = \nabla_\mu \delta n_\nu - \delta\Gamma^\lambda_{\mu\nu} n_\lambda \;\Rightarrow\; n_\lambda \delta\Gamma^\lambda_{\mu\nu} = \nabla_\mu \delta n_\nu - \delta(\nabla_\mu n_\nu) \,, \tag{2.21}$$

$$\delta(\nabla_\mu n^\mu) = \nabla_\mu \delta n^\mu + \delta\Gamma^\mu_{\mu\lambda} n^\lambda \;\Rightarrow\; n^\lambda \delta\Gamma^\mu_{\mu\lambda} = -\nabla_\mu \delta n^\mu + \delta(\nabla_\mu n^\mu)\,. \tag{2.22}$$

We can now simplify the argument in the boundary term as follows

$$\begin{aligned} n_\mu V^\mu &= g^{\alpha\beta} n_\mu \delta\Gamma^\mu_{\alpha\beta} - n^\alpha \delta\Gamma^\beta_{\beta\alpha} = \\ &= g^{\alpha\beta}(\nabla_\alpha \delta n_\beta - \delta(\nabla_\alpha n_\beta)) + \nabla_\alpha \delta n^\alpha - \delta(\nabla_\alpha n^\alpha) = \\ &= \nabla_\alpha(g^{\alpha\beta}\delta n_\beta) - \delta(g^{\alpha\beta}\nabla_\alpha n_\beta) + \delta g^{\alpha\beta}\nabla_\alpha n_\beta + \nabla_\alpha \delta n^\alpha - \delta(\nabla_\alpha n^\alpha) = \\ &= \nabla_\alpha(\delta n^\alpha + g^{\alpha\beta}\delta n_\beta) - 2\delta(\nabla_\alpha n^\alpha) + \nabla_\alpha n_\beta \delta g^{\alpha\beta} = \\ &\equiv \nabla_\alpha \delta u^\alpha - 2\delta(\nabla_\alpha n^\alpha) + \nabla_\alpha n_\beta \delta g^{\alpha\beta} \end{aligned} \tag{2.23}$$

where in the first equality we used the definition of $V^\mu$, in the second equality eq. (2.21) and eq. (2.22), in the third one the identity $\nabla_\alpha g^{\alpha\beta} = 0$ and in the last one we defined $\delta u^\alpha := \delta n^\alpha + g^{\alpha\beta}\delta n_\beta$. The vector $\delta u^\alpha$ lies on the boundary $\partial\mathcal{M}$, this can be shown as follows

$$\delta u^\alpha n_\alpha = n_\alpha \delta n^\alpha + n^\alpha \delta n_\alpha = \delta(n^\alpha n_\alpha) = 0\,. \tag{2.24}$$

We now calculate the first term of eq. (2.23), i.e. the four divergence $\nabla_\alpha \delta u^\alpha$. Since $\delta u^\alpha$ lies on $\partial\mathcal{M}$, we can use the relation of eq. (2.15) and get

$$\nabla_\alpha \delta u^\alpha = D_\alpha \delta u^\alpha - \varepsilon a_\beta \delta u^\beta\,, \tag{2.25}$$

where $a_\beta = n^\mu \nabla_\mu n_\beta$ and $D_\alpha$ is the induced covariant derivative on $\partial\mathcal{M}$ defined in eq. (2.12).

The second term in eq. (2.25) can be rewritten using

$$a_\beta \delta u^\beta = a_\beta \delta n^\beta + a^\beta \delta n_\beta = a_\beta n_\alpha \delta g^{\alpha\beta} + a^\alpha \delta n_\alpha = a_\beta n_\alpha \delta g^{\alpha\beta}\,, \tag{2.26}$$

where we used the definition of $\delta u^\alpha$ and the identity $a^\beta \delta n_\beta = 0$ due to the fact that $a_\beta$ and $n_\alpha$ are orthogonal because using eq. (2.6) one can show that $\delta n_\alpha = C(x^\mu) n_\alpha$ where $C(x^\mu)$ is a scalar function. Substituting this in eq. (2.25), gets us

$$\nabla_\alpha \delta u^\alpha = D_\alpha \delta u^\alpha - \varepsilon n_\alpha a_\beta \delta g^{\alpha\beta}\,. \tag{2.27}$$

We can now go back to eq. (2.23) and obtain

$$n_\mu V^\mu = D_\alpha \delta u^\alpha - 2\delta(\nabla_\alpha n^\alpha) + (\nabla_\alpha n_\beta - \varepsilon n_\alpha a_\beta)\delta g^{\alpha\beta}\,. \tag{2.28}$$

The quantity that emerged from the calculation turns out to be the *extrinsic curvature* tensor

$$K_{\alpha\beta} \equiv \nabla_\alpha n_\beta - \varepsilon n_\alpha a_\beta\,, \tag{2.29}$$

that has the following properties (without proof)

$$K_{\alpha\beta} = K_{\beta\alpha}\,,\;\; n^\alpha K_{\alpha\beta} = n^\beta K_{\alpha\beta} = 0\,,\;\; K = \nabla_\alpha n^\alpha\,. \tag{2.30}$$

Using eq. (2.9), the second property allows us to write $K_{\alpha\beta}\delta g^{\alpha\beta} = K_{\alpha\beta}\delta h^{\alpha\beta}$ and thus

$$\begin{aligned} \sqrt{|h|} n_\mu V^\mu &= \sqrt{|h|} D_\alpha \delta u^\alpha - 2\sqrt{|h|}\delta K + \sqrt{|h|} K_{\alpha\beta}\delta h^{\alpha\beta} = \\ &= \sqrt{|h|} D_\alpha \delta u^\alpha - \delta(2\sqrt{|h|}K) - \sqrt{|h|}(K h_{\alpha\beta} - K_{\alpha\beta})\delta h^{\alpha\beta}\,, \end{aligned} \tag{2.31}$$

where we used the well known relation for the variation of the square root of the determinant of the metric, i.e. $\delta\sqrt{|h|} = -\frac{1}{2}\sqrt{|h|}h_{\alpha\beta}\delta h^{\alpha\beta}$.

We can indeed rewrite eq. (2.20) as we anticipated in eq. (2.3) as

$$\begin{aligned} 16\pi\,\delta S_{EH} = &\int_{\mathcal{M}} d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} - \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}(Kh_{\alpha\beta} - K_{\alpha\beta})\delta h^{\alpha\beta} \\ &- \int_{\partial\mathcal{M}} d^3y\,\varepsilon\delta(2\sqrt{|h|}K) + \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}D_\alpha\delta u^\alpha\,. \end{aligned} \tag{2.32}$$

At this point one cannot fix all the boundary terms equal to zero, because 12 boundary conditions will be implicitly chosen: 6 from the $\delta h^{\alpha\beta}$ term and 6 from the $\delta(K\sqrt{|h|})$ term, making the theory overdetermined. Instead, what Gibbons, Hawking and York did was to fix the induced metric $h^{\alpha\beta}$ on the boundary and use the boundary term

$$S_{GHY} := \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}\,2K \tag{2.33}$$

as counter-term to correct the Einstein-Hilbert action. Note that by imposing

$$\delta h^{\alpha\beta}|_{\partial\mathcal{M}} = 0 \tag{2.34}$$

we get the natural boundary conditions

$$(Kh_{\alpha\beta} - K_{\alpha\beta})|_{\partial\mathcal{M}} = 0\,, \tag{2.35}$$

and taking the trace of this expression gets us to $K|_{\partial\mathcal{M}} = 0$. Therefore the natural boundary conditions is equivalent to

$$K_{\alpha\beta}|_{\partial\mathcal{M}} = 0\,. \tag{2.36}$$

The natural boundary conditions demands that the extrinsic curvature must be zero everywhere on the boundary surface. Using Stokes' theorem, the last term in eq. (2.32) can be converted into a boundary term on the two dimensional boundary $\partial^2\mathcal{M}$ and is usually ignored. The variational problem becomes well posed because

$$16\pi\,\delta S \equiv 16\pi\,\delta(S_{EH} + S_{GHY}) = \int_{\mathcal{M}} d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}\,, \tag{2.37}$$

delivers the Einstein field equations. Furthermore, by integrating this result we obtain the gravitational action for spacetime manifold with non-null boundaries up to a constant of integration that does not depend on the metric $g^{\mu\nu}$, i.e. $\delta S_0 = 0$,

$$S = S_{EH} + S_{GHY} + S_0\,. \tag{2.38}$$

### 2.2.4 Making the action physical

In the previous section we derived the gravitational action for spacetime manifolds with non-null boundaries up to a constant of integration $S_0$. Here we investigate the meaning of this term as done by E. Poisson [10] in his book. Note that $S_0$ does not play any physical contribution to the theory since it cannot affect the equation of motion.

To understand the role of $S_0$, we will calculate the gravitational action for flat spacetime. For the time being, let us assume that $S_0 = 0$ and let $g_{\mu\nu}$ be a solution of the *vacuum* Einstein field equation, thus $R = 0$ and therefore $S_{EH} = 0$, leaving us with

$$S = S_{GHY} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3y \, \varepsilon \sqrt{|h|} K \, . \tag{2.39}$$

The spacetime manifold can be understood as a collection of foliations, i.e. $\mathcal{M} = \bigcup_{t\in\mathbb{R}} \Sigma_t$, where the slice of the foliation $\Sigma_{t,\, t\in\mathbb{R}}$ is an hypersurface with the time component constant. We then choose $\partial\mathcal{M}$ to consist of two hypersurfaces $\Sigma_t$ with $t =$ const. and a three-cylinder $\mathcal{B}$ with radius $r = r_0$. Keeping in mind that the normal to $\partial\mathcal{M}$ must point outwards we have

$$\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2} \, , \tag{2.40}$$

where the minus sign in front of $\Sigma_{t_1}$ reminds us to correct the direction of the normal that point inwards the hypersurface $\partial\mathcal{M}$. See figure 4.1 on page 47 for a drawing of the spacetime foliation.

The two hypersurfaces of constant time are defined by $\Phi = t - t_i$ where $i = 1, 2$ and their induced metric can be read out from their line element $ds^2 = dx^2 + dy^2 + dz^2$. The unit normal has components $n_\alpha = \varepsilon \partial_\alpha \Phi = (-1, 0, 0, 0)$ since $\varepsilon = -1$ and the trace of the extrinsic curvature tensor trivially vanish $K = 0$ because all Christoffel symbols vanish.

The three-cylinder is defined by $\Psi = r - r_0$ and the induced metric $h_{\alpha\beta}$ is given by the line element $ds^2 = -dt^2 + r_0^2 d\Omega^2$ where we used spherical coordinates. One can then easily find $\sqrt{|h|} = r_0^2 \sin\theta$. The unit normal has components $n_\mu = \varepsilon \partial_\mu \Psi = (0, 1, 0, 0)$ where $\varepsilon = +1$ and we recover

$$\begin{aligned} K = \nabla_\mu n^\mu = \partial_\mu n^\mu + \Gamma^\mu_{\mu\lambda} n^\lambda = \Gamma^\mu_{\mu r} n^r &= \frac{1}{\sqrt{|h|}} \partial_r \sqrt{|h|} \bigg|_{r=r_0} = \\ &= \frac{1}{r^2 \sin\theta} \partial_r r^2 \sin\theta \bigg|_{r=r_0} = \frac{2}{r_0} \, . \end{aligned} \tag{2.41}$$

We then have

$$S = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3y \, \varepsilon \sqrt{|h|} K = \frac{1}{8\pi} \int_{t_1}^{t_2} dt \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, r_0^2 \sin\theta \frac{2}{r_0} = r_0 (t_2 - t_1) \, , \tag{2.42}$$

which diverges when the spacial boundary is pushed at infinity, i.e. $r_0 \to \infty$.

We have therefore shown that the gravitational action of a flat spacetime is infinite even when the spacetime manifold $\mathcal{M}$ is bounded by two hypersurfaces of constant time. This problem is there also when the spacetime is curved [10], making the gravitational action not a well-defined quantity for asymptotically-flat spacetimes.

We can solve this problem defining $S_0$ to be equal to the gravitational action of flat spacetime

$$S_0 = -\frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3y \, \varepsilon \sqrt{|h|} K_0 \, , \tag{2.43}$$

where $K_0$ is the extrinsic curature of $\partial\mathcal{M}$ embedded in flat spacetime. The minus sign in eq. (2.43) makes the quantity $S_{GHY} + S_0$ for $r_0 \to \infty$ well-defined.

## 2.3 Counter-term for null boundaries

### 2.3.1 Introduction

In the previous section we showed that the addition of the Gibbons-Hawking-York counter-term to the Einstein-Hilbert action makes the variational problem of general relativity well-posed. This result cannot be directly generalized to null boundaries because the normal to a null surface has zero norm and the three metric on a null surface is degenerate.

There are two possible ways to solve this problem. One possible approach would be to treat the null surface as the limit of a sequence of non-null surfaces. For example, one could perform the calculations on a timelike surface infinitesimally separated from the null surface and then take the limit. A more elegant approach would be to develop a procedure that is based on the properties of the null surface. For the non-null case, we obtain directly from the variational principle itself what to fix on the boundary. It is therefore desirable to develop a procedure that tell us what to fix on the non-null boundary to make the variational problem well-posed and not the other way around.

Recently, K. Parattu *et al.* [8] published the derivation of the counter-term for null boundaries starting from the variation of the Einstein-Hilbert action. After introducing the mathematical tools required for this derivation, we follow Parattu's approach as done in appendix G of [8] where he derives the counter-term for a general normal vector to the null-surface.

### 2.3.2 Mathematical framework

The unit normal to the null surface cannot be defined as we did for non-null surfaces because the normal to the null surface has a zero norm which makes the definition (2.6) undefined. We go around this problem defining the normal on the null-surface as

$$l_\alpha = A\Phi_{,\alpha} \tag{2.44}$$

where $\Phi$ is the scalar function that describe the null-surface and $A \neq 0$ is an arbitrary normalization scalar factor which may depend on the metric. Note that the choice of $l^\alpha$ is not unique because $A$ is not unique, unlike in the non-null case.

The second problem we are faceting when we try to generalize the non-null procedure is that the induced metric on a null surface is degenerate, what follows highlights this problem. One may try to use a definition analogous to the non-null case to define the induced metric

$$\widetilde{h}_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta \,, \tag{2.45}$$

but we run against a problem, namely, this metric is degenerate making it not well-defined. To see this, we can try to project $l^\alpha$ onto the null-surface and see that it does not vanish: $\widetilde{h}_{\alpha\beta} l^\alpha = l_\beta - \varepsilon l^\alpha l_\alpha l_\beta = l_\beta \neq 0$ where we used $l^\alpha l_\alpha = 0$ on the null surface. Since there is no straightforward extension of the non-null induced metric to the null one, we should dig deeper and find another way around this problem.

Luckily for us this problem was first solved by B. Carter [7]. We are going to follow his steps but we stick to the convention of K. Parattu such that the results found here can be used in the derivation of the counter-term. What is usually done is to introduce an auxiliary vector $k^\alpha$ such that $l_\alpha k^\alpha = -1$ holds everywhere.

Our first guess to find a projector to the null surface with the newly introduced auxiliary vector $k^\alpha$ is

$$\Pi^\alpha{}_\beta = \delta^\alpha{}_\beta + k^\alpha l_\beta \,, \tag{2.46}$$

which satisfies the projector condition $\Pi^\alpha{}_\beta \Pi^\beta{}_\gamma = \Pi^\alpha{}_\gamma$ and, in contrast to eq. (2.45), we have $\Pi^\alpha{}_\beta l_\alpha = 0$, but we still have $\Pi^\alpha{}_\beta l^\beta = l^\alpha \neq 0$. Also note that since $\Pi_{\alpha\beta} = g_{\alpha\beta} + k_\alpha l_\beta$ is not symmetric, it cannot be the induced metric we are looking for. We can easily define a new object which satisfy the symmetry condition, namely

$$q_{\alpha\beta} := g_{\alpha\beta} + k_\alpha l_\beta + l_\alpha k_\beta = \Pi_{\alpha\beta} + l_\alpha k_\beta \,. \tag{2.47}$$

We now demand that the projector $q^\alpha{}_\beta = \Pi^\alpha{}_\beta + l^\alpha k_\beta$ satisfies the projection condition

$$q^\alpha{}_\beta q^\beta{}_\gamma = \Pi^\alpha{}_\beta \Pi^\beta{}_\gamma + \Pi^\alpha{}_\beta l^\beta k_\gamma + l^\alpha k_\beta \Pi^\beta{}_\gamma + l^\alpha k_\beta l^\beta k_\gamma = q^\alpha{}_\gamma + l^\alpha k_\beta q^\beta{}_\gamma \stackrel{!}{=} q^\alpha{}_\beta \,, \tag{2.48}$$

which requires $l^\alpha k_\beta q^\beta{}_\gamma = 0$ and this is equivalent to

$$k_\beta q^\beta{}_\gamma = k_\gamma + k_\beta k^\beta l_\gamma - k_\gamma = 0 \;\Leftrightarrow\; k_\beta k^\beta = 0 \,, \tag{2.49}$$

which means we need $k^\alpha$ to be a null vector. Note that the projector $q^\alpha{}_\beta$ has the properties to project on the space orthogonal to $l^\alpha$, i.e. the null surface, since $q^\alpha{}_\beta l_\alpha = 0$ and $q^\alpha{}_\beta l^\beta = 0$ and on the space orthogonal to $k^\alpha$, since $q^\alpha{}_\beta k_\alpha = 0$ and $q^\alpha{}_\beta k^\beta = 0$. We are therefore further projecting the three null surface, orthogonal to $l^\alpha$, onto a two surface orthogonal to $k^\alpha$.

We now discuss the coordinate system on the null surface following Parattu's notation. As stated in Parattu's paper: any set of three continuous, infinitely differentiable functions, $y^a = (y^1, y^2, y^3)$ of the spacetime coordinates $x^\alpha$ constitutes a system of coordinates on the null surface if the set of values of these functions at every point on the null surface is unique. Then, the coordinate basis is the set of three vectors $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$. Let the parameter $\lambda$ vary smoothly on the null generators such that the displacements along the generators are of the form $dx^\alpha = l^\alpha d\lambda$. Ensuring that $\lambda$ varies smoothly for displacements across geodesics we can chose it to be one of the coordinates on the null surface. The other two coordinate are to be chosen as two smooth functions $z^A = (z^1, z^2)$ that are constant on each null geodesic. The basis vectors in the coordinate system $y^a = (\lambda, z^A)$ is

$$e^a_\lambda = \frac{\partial x^a}{\partial \lambda} \,,\; e^a_A = \frac{\partial x^a}{\partial z^A} \,,\; A = 1, 2 \,. \tag{2.50}$$

Note that $e^\alpha_A$ is a vector tangent to the two hypersurface, i.e. $l_\alpha e^\alpha_A = 0$ and $k_\alpha e^\alpha_A = 0$. Furthermore, the two linearly independent vectors $e^\alpha_A$ span the two dimensional hypersurface.

We can now express the induced metric $h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$ for a null surface using this coordinate basis. His components are $h_{\lambda\lambda} = g_{\alpha\beta} l^\alpha l^\beta = 0$, $g_{\lambda A} = g_{\alpha\beta} l^\alpha e^\beta_A = 0$ and $q_{AB} := g_{\alpha\beta} e^\alpha_A e^\beta_B$. Note that the determinant of the induced metric is zero and the line element is two dimensional

$$\begin{aligned} ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta} \left( \frac{\partial x^\alpha}{\partial z^A} dz^A \right) \left( \frac{\partial x^\beta}{\partial z^B} dz^B \right) = \\ = q_{\alpha\beta} e^\alpha_A e^\beta_B dz^A dz^B \equiv q_{AB} dz^A dz^B \,. \end{aligned} \tag{2.51}$$

The directed surface element for the null surface, see appendix A.3.4 of Parattu's paper [8] for the proof, is

$$d\Sigma_\alpha = \frac{\sqrt{-g}}{A} l_\alpha d^3 y = \frac{\sqrt{q}}{A} l_\alpha d\lambda d^2 z \tag{2.52}$$

where $q$ is the determinant of the two-matrix $q_{AB}$. Note that E. Poisson [10] uses a different convention for $l^\alpha$ and therefore the result of eq. (2.52) differs by a minus sign with the one of his book.

In the remaining part of this section we are going to introduce some definitions that are going to simplify the terms in the derivation of the counter term. The first one we want to introduce is the *second fundamental form* $\Theta_{\alpha\beta}$, also known as *extrinsic curvature*, for the null surface at any point we have

$$\Theta_{\alpha\beta} := \Pi^\gamma{}_\alpha \Pi^\delta{}_\beta \nabla_\gamma l_\delta = q^\gamma{}_\alpha q^\delta{}_\beta \nabla_\gamma l_\delta \,, \tag{2.53}$$

the proof of the last equality can be found in the appendix A.3.5 of Parattu's paper [8]. The trace of $\Theta_{\alpha\beta}$ is $\Theta = g^{\alpha\beta}\Theta_{\alpha\beta} = q^{\alpha\beta}\Theta_{\alpha\beta}$ and is known as the *expansion scalar*. It is the expansion along $l^\alpha$ of the two surface on the null surface orthogonal to $k^\alpha$.

Another useful object will be the *non-affinity coefficient* $\kappa$. We start by performing the manipulation of the following quantity

$$\begin{aligned} l^\alpha \nabla_\alpha l_\beta &= l^\alpha \nabla_\alpha (A \partial_\beta \Phi) = l^\alpha \frac{l_\beta}{A} \partial_\alpha A + l^\alpha A \nabla_\alpha \nabla_\beta \Phi = \\ &= l^\alpha \frac{l_\beta}{A} \partial_\alpha A + l^\alpha A \nabla_\beta \left( \frac{l_\alpha}{A} \right) = l^\alpha l_\beta \partial_\alpha \ln A + l^\alpha \nabla_\beta l_\alpha - l^\alpha l_\alpha \frac{1}{A} \nabla_\beta A = \\ &= (l^\alpha \partial_\alpha \ln A) l_\beta + \frac{1}{2} \partial_\beta (l^\alpha l_\alpha) \,, \end{aligned} \tag{2.54}$$

where in the third equality we have commuted the covariant derivatives and in the last equality we used $l^\alpha l_\alpha = 0$ on the null surface. The second term of eq. (2.54) can be rewritten using the following relation

$$\begin{aligned} \partial_\beta (l^\alpha l_\alpha) &= g_{\beta\gamma} \partial^\gamma (l^\alpha l_\alpha) = (q_{\beta\gamma} - k_\beta l_\gamma - l_\beta k_\gamma) \partial^\gamma (l^\alpha l_\alpha) = \\ &= q_{\beta\gamma} \partial^\gamma (l^\alpha l_\alpha) - k_\beta l_\gamma \partial^\gamma (l^\alpha l_\alpha) - l_\beta k_\gamma \partial^\gamma (l^\alpha l_\alpha) = \\ &= -k^\gamma \partial_\gamma (l^\alpha l_\alpha) l_\beta \,, \end{aligned} \tag{2.55}$$

where in the last equality we used that the first two terms vanish. The first one is zero because $q_{\beta\gamma}\partial^\gamma(l^\alpha l_\alpha) = q^\gamma{}_\beta \partial_\gamma (l^\alpha l_\alpha) = q^0{}_\beta \partial_0 (l^\alpha l_\alpha) \sim q^\alpha{}_\beta l_\alpha = 0$ since only[4] the component $x^0$ has $\partial_0(l^\alpha l_\alpha) \neq 0$ and the second term is zero because $l_\gamma \partial^\gamma (l^\alpha l_\alpha)$ is a derivative along the null surface and there $l^\alpha l_\alpha = 0$. We can therefore rewrite eq. (2.54) using the result of eq. (2.55) as follows

$$l^\alpha \nabla_\alpha l_\beta = \Big( l^\alpha \partial_\alpha \ln A - \frac{1}{2} k^\gamma \partial_\gamma (l^\alpha l_\alpha) \Big) l_\beta \,. \tag{2.56}$$

The definition $\kappa l_\beta := l^\alpha \nabla_\alpha l_\beta$ allow us to obtain the following relation for the non-affinity coefficient

$$\kappa = l^\alpha \partial_\alpha \ln A - \frac{1}{2} k^\gamma \partial_\gamma (l^\alpha l_\alpha) \,. \tag{2.57}$$

[4] To see this, consider a coordinate system $x^\alpha = (\Phi, x^1, x^2, x^3)$ where the first component describes the null surface. In such a coordinate system $l_\alpha = (A, 0, 0, 0)$ and only the $x^0$ component has $\partial_\beta (l^\alpha l_\alpha) \neq 0$.

If we now define $\widetilde{\kappa} := -\frac{1}{2}k^\gamma \partial_\gamma (l^\alpha l_\alpha) \equiv k^\alpha \partial_\alpha l^2$, then

$$\kappa - \widetilde{\kappa} = l^\alpha \partial_\alpha \ln A \,. \tag{2.58}$$

Finally, we derive a relation between the expansion scalar $\Theta$ and the non-affinity coefficient $\kappa$ that will be useful later on. The following expression can be rewritten as follows

$$\begin{aligned}\Theta + \kappa &= q^{\alpha\beta}\nabla_\alpha l_\beta - k^\gamma l_\gamma \kappa = \\ &= q^\alpha{}_\beta \nabla_\alpha l^\beta - k_\beta g^{\beta\gamma} l^\alpha \nabla_\alpha l_\gamma = \\ &= q^\alpha{}_\beta \nabla_\alpha l^\beta - k_\beta l^\alpha \nabla_\alpha l^\beta = \\ &= \Pi^\alpha{}_\beta \nabla_\alpha l^\beta \,,\end{aligned} \tag{2.59}$$

where in the third equality we used $\nabla_\alpha g^{\mu\nu} = 0$.

### 2.3.3 Derivation of the null counter-term

As we did for the non-null counter-term, we start with the well known result

$$\begin{aligned}16\pi\, \delta S_{EH} &= \int_{\mathcal{M}} d^4x\, \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} + \int_{\mathcal{M}} d^4x\, \sqrt{-g} \nabla_\mu V^\mu = \\ &= \int_{\mathcal{M}} d^4x\, \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^3y\, \frac{\sqrt{-g}}{A} l_\mu V^\mu \,,\end{aligned} \tag{2.60}$$

where $V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\beta\alpha}$. Once again we used Gauss' theorem to get from the first to the second expression. Performing the same manipulations on the argument of the boundary term as we did in the non-null case to arrive at eq. (2.23), we find

$$l_\mu V^\mu = \nabla_\alpha \delta u^\alpha - 2\delta(\nabla_\alpha l^\alpha) + \nabla_\alpha l_\beta \delta g^{\alpha\beta} \,, \tag{2.61}$$

where $\delta u^\alpha := \delta l^\alpha + g^{\alpha\beta}\delta l_\beta$. This relation can be rewritten using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}$ as

$$\frac{\sqrt{-g}}{A} l_\mu V^\mu = \frac{1}{A}\Big(\sqrt{-g}\nabla_\alpha \delta u^\alpha - 2\delta(\sqrt{-g}\nabla_\alpha l^\alpha) + \sqrt{-g}(\nabla_\alpha l_\beta - g_{\alpha\beta}\nabla_\lambda l^\lambda)\delta g^{\alpha\beta}\Big) \,. \tag{2.62}$$

The first thing we want to do is to separate out the surface term from the *first* term in eq. (2.62). Using the following relation $\nabla_\mu v^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|} v^\mu)$ to calculate the divergence of a vector field $v^\mu$, we obtain

$$\frac{\sqrt{-g}}{A}\nabla_\alpha \delta u^\alpha = \frac{1}{A}\partial_\alpha(\sqrt{-g}\delta u^\alpha) = \partial_\alpha\left(\frac{\sqrt{-g}}{A}\delta u^\alpha\right) - \sqrt{-g}\delta u^\alpha \partial_\alpha\left(\frac{1}{A}\right) \,, \tag{2.63}$$

where we can isolate the derivatives along the null-surface from the first term of eq. (2.63). Using the projector $\Pi^\alpha{}_\beta$ given in eq. (2.46), we proceed as follows

$$\begin{aligned}\partial_\alpha\left(\frac{\sqrt{-g}}{A}\delta u^\alpha\right) &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta \delta u^\beta\right) - \partial_\alpha\left(\frac{\sqrt{-g}}{A}k^\alpha l_\beta \delta u^\beta\right) = \\ &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta \delta u^\beta\right) - \frac{\sqrt{-g}}{A}\delta(l_\beta l^\beta)\partial_\alpha k^\alpha - \frac{\sqrt{-g}}{A}k^\alpha \partial_\alpha(\delta(l_\beta l^\beta)) = \\ &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta \delta u^\beta\right) - \frac{\sqrt{-g}}{A}k^\alpha \partial_\alpha(\delta(l_\beta l^\beta)) \,,\end{aligned} \tag{2.64}$$

where in the second equality we used $l_\beta \delta u^\beta = l_\beta \delta l^\beta + l_\beta g^{\beta\alpha}\delta l_\alpha = 2 l_\beta \delta l^\beta = \delta(l_\beta l^\beta)$ and in the last one $\delta(l_\beta l^\beta) = 0$ on the null-surface. The first term in eq. (2.64) is a surface derivative on the null surface and the second term contains variation of the derivatives of the metric. We can further simplify the second term of eq. (2.64) as

$$-k^\alpha \partial_\alpha(\delta(l_\beta l^\beta)) = -\delta(k^\alpha \partial_\alpha(l_\beta l^\beta)) + \delta k^\alpha \partial_\alpha(l_\beta l^\beta)\,. \tag{2.65}$$

Summarizing all these results, the first term of eq. (2.62) becomes

$$\begin{aligned}\frac{\sqrt{-g}}{A}\nabla_\alpha \delta u^\alpha &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta \delta u^\beta\right) - \frac{\sqrt{-g}}{A}\delta(k^\alpha\partial_\alpha(l_\beta l^\beta)) + \frac{\sqrt{-g}}{A}\delta k^\alpha \partial_\alpha(l_\beta l^\beta) - \sqrt{-g}\delta u^\alpha \partial_\alpha\left(\frac{1}{A}\right) = \\ &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta \delta u^\beta\right) - \delta\left(\frac{\sqrt{-g}}{A}k^\alpha\partial_\alpha(l_\beta l^\beta)\right) - \frac{\sqrt{-g}}{2A}\left(k^\alpha\partial_\alpha(l_\beta l^\beta)\right) g_{\mu\nu}\delta g^{\mu\nu} \\ &\quad + \sqrt{-g}k^\alpha\partial_\alpha(l_\beta l^\beta)\delta\left(\frac{1}{A}\right) + \frac{\sqrt{-g}}{A}\delta k^\alpha\partial_\alpha(l_\beta l^\beta) - \sqrt{-g}\delta u^\alpha\partial_\alpha\left(\frac{1}{A}\right)\,,\end{aligned} \tag{2.66}$$

where in the second equality we used once again $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Note that all the variations of the derivatives of the metric are in the first two terms of eq. (2.66), this is true only if the factor $A$ does not depend on the derivative of the metric.

The *second* term of eq. (2.62) can be rewritten as

$$-\frac{2}{A}\delta(\sqrt{-g}\nabla_\alpha l^\alpha) = -2\delta(\frac{\sqrt{-g}}{A}\nabla_\alpha l^\alpha) + 2\sqrt{-g}\nabla_\alpha l^\alpha \delta\left(\frac{1}{A}\right)\,. \tag{2.67}$$

We now want to substitute back the terms of eq. (2.66) and eq. (2.67) in eq. (2.62). To simplify the expression we use the following relation in three different places

$$\nabla_\alpha l^\alpha + \frac{k^\alpha}{2}\partial_\alpha(l_\beta l^\beta) = \delta^\alpha{}_\beta \nabla_\alpha l^\beta + k^\alpha l_\beta \nabla_\alpha l^\beta = \Pi^\alpha{}_\beta\nabla_\alpha l^\beta\,, \tag{2.68}$$

where we used $\partial_\alpha(l_\beta l^\beta) = \nabla_\alpha(l_\beta l^\beta) = l_\beta\nabla_\alpha l^\beta + l^\beta\nabla_\alpha l_\beta = 2l_\beta\nabla_\alpha l^\beta$ and the definition of $\Pi^\alpha{}_\beta$. Thus eq. (2.62), the argument of the boundary element on the null surface, becomes

$$\begin{aligned}\frac{\sqrt{-g}}{A}l_\mu V^\mu &= \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta\delta u^\beta\right) - 2\delta\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta\nabla_\alpha l^\beta\right) \\ &\quad + \frac{\sqrt{-g}}{A}(\nabla_\alpha l_\beta - g_{\alpha\beta}\Pi^\gamma{}_\lambda\nabla_\gamma l^\lambda)\delta g^{\alpha\beta} + 2\sqrt{-g}\Pi^\alpha{}_\beta\nabla_\alpha l^\beta\delta\left(\frac{1}{A}\right) \\ &\quad + \frac{\sqrt{-g}}{A}\delta k^\alpha\partial_\alpha(l_\beta l^\beta) - \sqrt{-g}\delta u^\alpha\partial_\alpha\left(\frac{1}{A}\right)\,.\end{aligned} \tag{2.69}$$

The second term of eq. (2.69) can be rewritten using the projector $q^\alpha{}_\beta = \Pi^\alpha{}_\beta + l^\alpha k_\beta$. Using eq. (2.59) we get

$$2\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta\nabla_\alpha l^\beta = 2\frac{\sqrt{-g}}{A}(\Theta + \kappa)\,, \tag{2.70}$$

where $\Theta$ is the expansion scalar and $\kappa$ the non-affinity coefficient.

As we mentioned in the previous section we have $l_\alpha l^\alpha = 0$ on the null surface and we demand that $l_\alpha k^\alpha = -1$ and $k_\alpha k^\alpha = 0$ holds everywhere. Since we are interested in surfaces that stay null during the variation, the relations $q^{\alpha\beta}l_\alpha = 0$ and $q^{\alpha\beta}k_\alpha = 0$ holds even during the variation, i.e. $\delta q^{\alpha\beta}l_\alpha l_\beta = \delta q^{\alpha\beta}l_\alpha k_\beta = \delta q^{\alpha\beta}k_\alpha k_\beta = 0$. We use these relations to simplify the term $g_{\alpha\beta}\delta g^{\alpha\beta}$, which we will use to simplify the *third* term of eq. (2.69). We proceed as follows

$$\begin{aligned} g_{\alpha\beta}\delta g^{\alpha\beta} &= g_{\alpha\beta}(\delta q^{\alpha\beta} - \delta(k^\alpha l^\beta) - \delta(l^\alpha k^\beta)) = q_{\alpha\beta}\delta q^{\alpha\beta} + 2l_\alpha k_\beta \delta(k^\alpha l^\beta) + 2l_\alpha k_\beta \delta(l^\alpha k^\beta) = \\ &= q_{\alpha\beta}\delta q^{\alpha\beta} + 2l_\alpha k_\beta l^\beta \delta k^\alpha + 2k_\beta l_\alpha k^\alpha \delta l^\beta = q_{\alpha\beta}\delta q^{\alpha\beta} - 2\delta\ln A - 2k_\beta\delta l^\beta \end{aligned} \tag{2.71}$$

where in the third equality we used $l_\alpha k_\beta \delta(l^\alpha k^\beta) = k_\beta\delta(l_\alpha l^\alpha k^\beta) - k_\beta k^\beta l^\alpha \delta l_\alpha = 0$ and in the last equality $l_\alpha k^\alpha = -1$, $l_\alpha \delta k^\alpha = \delta \ln A$ obtained from $\delta(l_\alpha k^\alpha) = 0$ and $\delta l_\alpha = l_\alpha \delta \ln A$. Next, we need to simplify the second term of eq. (2.69), i.e. $(\nabla_\alpha l_\beta)\delta g^{\alpha\beta}$. To do so, we use the definition of the induced metric $q^{\alpha\beta}$ as follows

$$\begin{aligned} (\nabla_\alpha l_\beta)\delta g^{\alpha\beta} &= \nabla_\alpha l_\beta \delta q^{\alpha\beta} - \nabla_\alpha l_\beta \delta(k^\alpha l^\beta) - \nabla_\alpha l_\beta \delta(l^\alpha k^\beta) = \\ &= \nabla_\alpha l_\beta \delta q^{\alpha\beta} - \nabla_\alpha l_\beta \delta k^\alpha l^\beta - \nabla_\alpha l_\beta k^\alpha \delta l^\beta - \nabla_\alpha l_\beta \delta l^\alpha k^\beta - \nabla_\alpha l_\beta l^\alpha \delta k^\beta = \\ &= \nabla_\alpha l_\beta \delta q^{\alpha\beta} - \delta l^\alpha k^\beta(\nabla_\alpha l_\beta + \nabla_\beta l_\alpha) - l^\beta \nabla_\alpha l_\beta \delta k^\alpha - l^\alpha \nabla_\alpha l_\beta \delta k^\beta = \\ &= \nabla_\alpha l_\beta \delta q^{\alpha\beta} - \delta l^\alpha k^\beta(\nabla_\alpha l_\beta + \nabla_\beta l_\alpha) - \frac{1}{2}\partial_\beta l^2 \delta k^\beta - \kappa\delta\ln A\,, \end{aligned} \tag{2.72}$$

where in the last equality we used the definition of the non-affinity coefficient $\kappa l_\beta := l^\alpha \nabla_\alpha l_\beta$ together with $\delta(l_\alpha k^\alpha) = 0 \Leftrightarrow l_\alpha \delta k^\alpha = -\delta l_\alpha k^\alpha = -\delta \ln A\, l_\alpha k^\alpha = \delta \ln A$ and the short hand notation $\frac{1}{2}\partial_\alpha l^2 \equiv l^\beta \nabla_\alpha l_\beta$. The first term of eq. (2.72) can be further simplified

$$\begin{aligned} \nabla_\alpha l_\beta \delta q^{\alpha\beta} &= \delta^\mu{}_\alpha \delta^\nu{}_\beta \nabla_\mu l_\nu \delta q^{\alpha\beta} = \\ &= (q^\mu{}_\alpha - k^\mu l_\alpha - l^\mu k_\alpha)(q^\nu{}_\beta - k^\nu l_\beta - l^\nu k_\beta)\nabla_\mu l_\nu \delta q^{\alpha\beta} = \\ &= (q^\mu{}_\alpha - l^\mu k_\alpha)(q^\nu{}_\beta - l^\nu k_\beta)\nabla_\mu l_\nu \delta q^{\alpha\beta} = \\ &= (q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu l_\nu - q^\mu{}_\alpha l^\nu k_\beta \nabla_\mu l_\nu - q^\nu{}_\beta l^\mu k_\alpha \nabla_\mu l_\nu - l^\mu k_\alpha l^\nu k_\beta \nabla_\mu l_\nu)\delta q^{\alpha\beta} = \\ &= (\Theta_{\alpha\beta} - \frac{1}{2} q^\mu{}_\alpha k_\beta \partial_\mu l^2 - \kappa q^\nu{}_\beta k_\alpha l_\nu)\delta q^{\alpha\beta} = \\ &= \Theta_{\alpha\beta}\delta q^{\alpha\beta}\,, \end{aligned} \tag{2.73}$$

where in the third equality we used $l_\alpha \delta q^{\alpha\beta} = \delta(l_\alpha q^{\alpha\beta}) - q^{\alpha\beta}\delta l_\alpha = -q^{\alpha\beta} l_\alpha \delta \ln A = 0$, in the fifth one the definitions of $\Theta_{\alpha\beta}$ and $\kappa$ given in eqs. (2.53) and (2.54), $k_\alpha k_\beta \delta q^{\alpha\beta} = 0$ and finally, in the last one, we used $k_\alpha q^\nu{}_\beta l_\nu \delta q^{\alpha\beta} = k_\alpha l_\beta \delta q^{\alpha\beta} = 0$ and $q^{\mu\alpha}\partial_\mu l^2 = 0$ since it is a derivative on the null surface an there it holds $l^2 \equiv l_\alpha l^\alpha = 0$. Furthermore, the last two term of eq. (2.72) can be simplified as follows

$$\begin{aligned} -\frac{1}{2}\partial_\beta l^2 \delta k^\beta - \kappa\delta\ln A &= -\frac{1}{2}\partial_0 l^2 \delta k^0 - \kappa\delta\ln A = -\delta\ln A\left(\frac{\partial_0 l^2}{2A} + \kappa\right) = \\ &= -\delta \ln A\left(-\frac{k^\alpha \partial_\alpha l^2}{2} + \kappa\right) = -\delta\ln A(\widetilde{\kappa} + \kappa)\,, \end{aligned} \tag{2.74}$$

where in the first equality we used that only the derivative with respect to the $x^0$ coordinate is non-zero[5] on the null surface and in the last one the definition $\widetilde{\kappa} := -\frac{1}{2}k^\alpha \partial_\alpha l^2$.

[5] We are using a coordinate system $x^\alpha = (\Phi, \lambda, z^1, z^2)$ where the first component describes the null surface. In this coordinate system $l_\alpha = (A, 0, 0, 0)$ and only the component $x^0$ has $\partial_\beta(l^\alpha l_\alpha) \neq 0$.

Putting all these parts together we rewrite the third term of eq. (2.69) as

$$\frac{\sqrt{-g}}{A}(\nabla_\alpha l_\beta - g_{\alpha\beta}\Pi^\gamma{}_\lambda\nabla_\gamma l^\lambda)\delta g^{\alpha\beta} = \frac{\sqrt{-g}}{A}\left(\Theta_{\alpha\beta}\delta q^{\alpha\beta} - \delta l^\alpha k^\beta(\nabla_\alpha l_\beta + \nabla_\beta l_\alpha) - \delta\ln A(\kappa+\widetilde{\kappa})\right) \\ - (\Theta+\kappa)(q_{\alpha\beta}\delta q^{\alpha\beta} - 2\delta\ln A - 2k_\beta\delta l^\beta)\,. \tag{2.75}$$

The *fourth*, *fifth* and *sixth* term of eq. (2.69) can also be simplified as follows

$$2\sqrt{-g}\Pi^\alpha{}_\beta\nabla_\alpha l^\beta\delta\left(\frac{1}{A}\right) = -2\sqrt{-g}(\Theta+\kappa)\frac{\delta\ln A}{A}\,, \tag{2.76}$$

$$\frac{\sqrt{-g}}{A}\delta k^\alpha\partial_\alpha(l_\beta l^\beta) = \frac{\sqrt{-g}}{A}(-\delta\ln A)\,k^\alpha\partial_\alpha l^2 = 2\frac{\sqrt{-g}}{A}\widetilde{\kappa}\,\delta\ln A\,, \tag{2.77}$$

$$\begin{aligned} -\sqrt{-g}\delta u^\alpha\partial_\alpha\left(\frac{1}{A}\right) &= \sqrt{-g}(\delta l^\alpha + g^{\alpha\beta}\delta l_\beta)\frac{\partial_\alpha A}{A^2} = \\ &= \frac{\sqrt{-g}}{A}(\delta l^\alpha\partial_\alpha\ln A + g^{\alpha\beta}\delta l_\beta\partial_\alpha\ln A) = \\ &= \frac{\sqrt{-g}}{A}(\delta l^\alpha\partial_\alpha\ln A + l^\alpha\partial_\alpha\ln A\,\delta\ln A) = \\ &= \frac{\sqrt{-g}}{A}(\delta l^\alpha\partial_\alpha\ln A + (\kappa-\widetilde{\kappa})\delta\ln A)\,, \end{aligned} \tag{2.78}$$

where we used eq. (2.59) in the first term, $\delta k^\alpha = -k^\alpha\,\delta\ln A$ and $\widetilde{\kappa} = -\frac{1}{2}k^\alpha\partial_\alpha l^2$ in the second term and in the last one $\delta l_\beta = l_\beta\,\delta\ln A$ and eq. (2.58). Finally, we can insert all the simplified term in eq. (2.69) and obtain

$$\frac{\sqrt{-g}}{A}l_\mu V^\mu = \partial_\alpha\left(\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta\delta u^\beta\right) - 2\delta\left(\frac{\sqrt{-g}}{A}(\Theta+\kappa)\right) + \frac{\sqrt{-g}}{A}(\Theta_{\alpha\beta} - q_{\alpha\beta}(\Theta+\kappa))\delta q^{\alpha\beta} \\ + \frac{\sqrt{-g}}{A}\Big(2k_\alpha(\Theta+\kappa) - k^\beta(\nabla_\alpha l_\beta + \nabla_\beta l_\alpha) + \partial_\alpha\ln A\Big)\delta l^\alpha\,. \tag{2.79}$$

We now write the boundary term using the coordinate system $y^a = (\lambda, z^1, z^2)$ and the respective surface element given in eq. (2.52), i.e. $\frac{\sqrt{-g}}{A}d^3y = \frac{\sqrt{q}}{A}d\lambda d^2z$, thus

$$\begin{aligned} \int_{\partial\mathcal{M}} d^3y\,\frac{\sqrt{-g}}{A}l_\mu V^\mu = \int_{\partial\mathcal{M}} d\lambda d^2z\,\Bigg\{&\partial_\alpha\left(\frac{\sqrt{q}}{A}\Pi^\alpha{}_\beta\delta u^\beta\right) - 2\delta\left(\frac{\sqrt{q}}{A}(\Theta+\kappa)\right) \\ &+ \frac{\sqrt{q}}{A}(\Theta_{\alpha\beta} - q_{\alpha\beta}(\Theta+\kappa))\delta q^{\alpha\beta} \\ &+ \frac{\sqrt{q}}{A}\Big(2k_\alpha(\Theta+\kappa) - k^\beta(\nabla_\alpha l_\beta + \nabla_\beta l_\alpha) + \partial_\alpha\ln A\Big)\delta l^\alpha\Bigg\}\,. \end{aligned} \tag{2.80}$$

The terms in the first two lines of eq. (2.80) have the same structure as the one of the non-null case, namely, the first term is the three derivative that is usually ignored, the second one is the boundary counter-term and the third one is the one we will kill fixing $q^{\alpha\beta}$ on the boundary. In contrast to the non-null case here we have an extra term with $\delta l^\alpha$ which we have to deal with and make sure that we do not overdetermine the theory fixing $l^\alpha$ on the boundary surface. Setting $\delta q^{\alpha\beta} = 0$ on the boundary fixes 4-1=3 degrees of freedom (the minus one account for the symmetry condition $q_{\alpha\beta} = q_{\beta\alpha}$), while setting $\delta l^\alpha = 0$ on the boundary fixes other 4-1=3 degrees of freedom (the minus one account for the condition $l^\alpha l_\alpha = 0$). These 6 degrees of freedom are compensated by the 6 degrees of freedom of the counter-term used to correct the Einstein-Hilbert action, namely

$$S_{\text{null}} := \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3y\, 2\frac{\sqrt{-g}}{A}\Pi^\alpha{}_\beta\nabla_\alpha l^\beta = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d\lambda d^2z\, 2\frac{\sqrt{q}}{A}(\Theta+\kappa)\,. \tag{2.81}$$

If we fix the induced metric $q_{\alpha\beta}$ and the normal to the null surface $l^\alpha$ on the boundary, the corrected gravitational action gives the following variational problem

$$16\pi\delta S \equiv 16\pi\delta(S_{EH}+S_{\text{null}}) = \int_{\mathcal{M}} d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}\,. \tag{2.82}$$

By integrating we conclude that the gravitational action for null surfaces is given up to a functional $S_0$, independent of $g_{\mu\nu}$, as

$$S = S_{EH} + S_{\text{null}} + S_0\,. \tag{2.83}$$

The term $S_0$ plays the same role as the one in the non-null case, namely, it ensures that the action does not diverges for $r\to\infty$. To determinate $S_0$, one may follows the same procedure showed in section 2.2.4.

## 2.4 Boundary terms and the tetrad formalism

### 2.4.1 Introduction

Recently, following Parattu's publication, Jubb *et al.* [14] published a paper discussing the boundary and corner terms of the action of general relativity. Interesting to us, they showed a unification for the derivation of spacelike, timelike and null-like boundary terms using Cartan's tetrad formalism. As we will see, the use of the tetrad formulation of Einstein's theory simplify a lot the derivation of both boundary terms.

After a short introduction of the tetrad formalism, we show the equivalence between the tetrad Einstein-Hilbert action to the one of the conventional theory. Then, once the mathematical framework is set up, we follow Jubb's procedure and elegantly derive both boundary terms.

### 2.4.2 The tetrad formalism

Cartan's tetrad formalism is also known as Einstein's "vierbein" theory and dates back to Einstein's research of the 20s. As a formalism rather than a theory, it does not make any different prediction but allows to express the relations of general relativity in a useful different way. It was introduced to represent a relativistic quantum field theory in curved spacetime. Different authors, in the literature, use different notations which may cause some confusions. Here we will introduce the tetrad formalism following the convention[6] of J. Yepez [15].

The conventional approach to general relativity uses the "natural" differential basis, namely, the tangent space $T_p$ at point $p$ is spanned by a set of partial derivatives at that point

$$e_\mu = \partial_\mu \,, \tag{2.84}$$

while the cotangent space $T_p^*$ at point $p$ is spanned by a set of differential elements

$$e^\mu = dx^\mu \,, \tag{2.85}$$

which lie in the direction of the gradient of the coordinates functions. A four-vector $A \in T_p$ has components $A = A^\mu e_\mu = (A^0, A^1, A^2, A^3)$, while a dual four-vector $A \in T_p^*$ has components $A = A_\nu e^\nu = g_{\mu\nu} A^\nu e^\mu$. The partial derivatives and the differential elements are inverses of each others, i.e. $e^\mu \otimes e_\nu = 1^\mu{}_\nu$. One is allowed to choose any orthonormal basis to span $T_p$ as long as it has the signature of the manifold one is working on. We can then introduce a set of basis vectors $e_a$ as non-coordinate unit vectors, where we use small Latin letters to denote the indices, with inner product

$$\langle e_a, e_b \rangle = \eta_{ab} \,, \tag{2.86}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric of flat spacetime.

The *tetrad basis* is the orthonormal basis independent of the coordinates. Note that it is not possible to find a chart that cover the entire curved manifold, but we still can choose

[6]Yepez's convention differs from the one of S. Weinberg [3] and N. Straumann [2]. Here the tetrads $e_\mu{}^a$ have first the Greek index and second the Latin one. Furthermore, the notation of the covariant derivative $\nabla_\mu$ differs from the one of Weinberg and Straumann, see eq. (2.94) and the following remark.

a fixed orthonormal basis independent of the position. We will then express any vector at a point $p$ as a linear combination of the fixed tetrad basis vectors at that point. Denoting an element of the tetrad basis by $e_a$, the coordinate basis is expressed in terms of the tetrads as follows

$$e_\mu(x) = e_\mu{}^a(x) e_a \,, \tag{2.87}$$

where the functional components $e_\mu{}^a(x)$ form a $4 \times 4$ invertible matrix. The *tetrads* or *vierbeins* are the four objects $e_\mu{}^a$ where $a = 1, 2, 3, 4$. The inverse of the tetrad is $e^\mu{}_a$ and it allows to write $e_a = e^\mu{}_a e_\mu$. The tetrads satisfy the following identities

$$e^\mu{}_a(x) e_\nu{}^a(x) = \delta^\mu_\nu \ , \ e_\mu{}^a(x) e^\mu{}_b(x) = \delta^a_b \,. \tag{2.88}$$

Furthermore, we can use the metric $g_{\mu\nu}$ to induce the product of the tetrad fields and inverse tetrad fields, as follows

$$\eta_{ab} = g_{\mu\nu}(x) e^\mu{}_a(x) e^\nu{}_b(x) \,, \tag{2.89}$$

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x) \eta_{ab} \ . \tag{2.90}$$

One can now form a dual orthonormal basis using a set of one-forms $e^a \in T^*_p$ that satisfy $e^a \otimes e_b = 1^a{}_b$. Then, the non-coordinate basis can express as linear combination of the coordinate basis and vice versa

$$e^a(x) = e_\mu{}^a(x) e^\mu(x) \ , \ e^\mu(x) = e^\mu{}_a(x) e^a. \tag{2.91}$$

Any vector at any spacetime point can be represented using coordinate and non-coordinate orthonormal basis

$$\boldsymbol{V} = V^\mu e_\mu = V^a e_a \,, \tag{2.92}$$

and its components are $V^\mu = e^\mu{}_a V^a$ or $V^a = e_\mu{}^a V^\mu$. Note that in general the Greek indices are raised and lowered by the metric $g_{\mu\nu}$, while the Latin indices are raised and lowered by the Minkowski metric $\eta_{ab}$.

In the non-coordinate-based geometry, the affine connection coefficients $\Gamma^\lambda_{\mu\nu}$ are replaced by the spin connection coefficients $\omega_\mu{}^a{}_b$. One finds, see Yepez's paper [15] for the derivation, the following relation which express the spin connection in function of the affine connection

$$\omega_\mu{}^a{}_b = e_\kappa{}^a e^\lambda{}_b \Gamma^\kappa_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a \equiv -e^\lambda{}_b D_\mu e_\lambda{}^a = -e_{\lambda b} D_\mu e^{\lambda a} \,, \tag{2.93}$$

where we used $D_\lambda g_{\mu\nu} \equiv \nabla_\lambda g_{\mu\nu} = 0$ in the last equality. This relation can be used to obtain the *tetrad postulate*, i.e. $\nabla_\mu e_\nu{}^a = 0$, multiplying eq. (2.93) by $e_\nu{}^b$ and using the identity of eq. (2.88), we obtain

$$\begin{aligned} \omega_\mu{}^a{}_b e_\nu{}^b =& e_\kappa{}^a e^\lambda{}_b e_\nu{}^b \Gamma^\kappa_{\mu\lambda} - e^\lambda{}_b e_\nu{}^b \partial_\mu e_\lambda{}^a = e_\kappa{}^a \Gamma^\kappa_{\mu\nu} - \partial_\nu e_\nu{}^a \\ \Leftrightarrow& \ \nabla_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - \Gamma^\kappa_{\mu\nu} e_\kappa{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \,. \end{aligned} \tag{2.94}$$

Note that here we are not using the convention of Weinberg [3] or Straumann [2], they use the symbol of the covariant derivative $\nabla_\mu$ to indicate $D_\mu$, the first two terms of eq. (2.94). The covariant derivative of a coordinate vector and one-form are

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda \,, \tag{2.95}$$

$$\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\lambda_{\nu\mu} X_\lambda \,, \tag{2.96}$$

and similarly for the non-coordinate vector and the one-form

$$\nabla_\mu X^a = \partial_\mu X^a + \omega_\mu{}^a{}_b X^b \,, \tag{2.97}$$

$$\nabla_\mu X_a = \partial_\mu X_a - \omega_\mu{}^b{}_a X_b \,. \tag{2.98}$$

Furthermore, under Lorentz transformation the tetrads transform as a tensor, while the spin connection must transform as a connection

$$e_\mu{}^a \to \Lambda^a{}_c e_\mu{}^c \,\,,\,\, \omega_\mu{}^{ab} \to \Lambda^a{}_c \omega_\mu{}^{cd} (\Lambda^{-1})^b{}_d - (\Lambda^{-1})^a{}_c \partial_\mu \Lambda^c{}_b \,. \tag{2.99}$$

Now that the mathematical framework is set up, we introduce Cartan's notation which further simplify the tetrad formalism. As we saw, the non-coordinate basis one-form is $e^a = e_\mu{}^a e_\mu = e_\mu{}^a dx^\mu$, in analogy one writes

$$\omega^a{}_b = \omega_\mu{}^a{}_b dx^\mu \,. \tag{2.100}$$

Furthermore, we introduce the following two notations for the differential form and the wedge product of two vectors **A** and **B**

$$d\mathbf{A} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{2.101}$$

$$\mathbf{A} \wedge \mathbf{B} \equiv A_\mu B_\nu - A_\nu B_\mu \,, \tag{2.102}$$

which are both antisymmetric in the Greek indices. With this notation one can write the torsion tensor and the Riemann curvature as

$$T^a = de^a + \omega^a{}_b \wedge e^b \,, \tag{2.103}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,, \tag{2.104}$$

which is a compact notation for, see Yepez's paper [15] for the derivation,

$$T_{\mu\nu}{}^a = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b \,, \tag{2.105}$$

$$T_{\mu\nu}{}^\lambda = e^\lambda{}_a T_{\mu\nu}{}^a = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0 \,, \tag{2.106}$$

$$R_{\mu\nu}{}^a{}_b = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c\, \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c\, \omega_\mu{}^c{}_b \,, \tag{2.107}$$

$$R^\lambda{}_{\sigma\mu\nu} = e^\lambda{}_a e_\sigma{}^b R_{\mu\nu}{}^a{}_b = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\tau} \Gamma^\tau_{\nu\sigma} - \Gamma^\lambda_{\nu\tau} \Gamma^\tau_{\mu\sigma} \,, \tag{2.108}$$

Note that here we use Yepez's convention for the Riemann tensor which differs from our convention given in eq. (1.2).

Before proceeding with the derivation of the tetrad Einstein-Hilbert action, we derive a few relations that will be useful later on. One can easily find using eq. (2.104) and eq. (2.108) the following relations

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \,, \tag{2.109}$$

$$R_{\mu\nu}{}^{ab} = e^{\lambda a} e^{\kappa b} R_{\mu\nu\lambda\kappa} \,. \tag{2.110}$$

Then, using the skew and the interchange symmetries of the Riemann tensor $R_{\mu\nu\lambda\kappa}$ we find

$$R_{\mu\nu}{}^{ab} = -R_{\nu\mu}{}^{ab} = R_{\nu\mu}{}^{ba} \,. \tag{2.111}$$

Furthermore, using the relation given in eq. (2.93), one can find the antisymmetry property of ${\omega_\mu}^{ab}$ with respect to the Latin indices as follows

$$
{\omega_\mu}^{ab} = -{\omega_\mu}^{ba} \,. \tag{2.112}
$$

Finally, we compute the variation of the Riemann curvature tensor as follows

$$
\begin{aligned}
\delta R^{ab} &= \delta(d\omega^{ab}) + \delta({\omega^a}_c \wedge \omega^{cb}) = \\
&= \delta(\partial_\mu {\omega_\nu}^{ab} - \partial_\nu {\omega_\mu}^{ab}) + \delta({\omega_\mu}^a{}_c\, {\omega_\nu}^{cb} - {\omega_\nu}^a{}_c\, {\omega_\mu}^{cb}) = \\
&= \partial_\mu \delta {\omega_\nu}^{ab} - \partial_\nu \delta {\omega_\mu}^{ab} + \delta {\omega_\mu}^a{}_c\, {\omega_\nu}^{cb} + {\omega_\mu}^a{}_c\, \delta {\omega_\nu}^{cb} \\
&\quad - \delta {\omega_\nu}^a{}_c\, {\omega_\mu}^{cb} - {\omega_\nu}^a{}_c\, \delta {\omega_\mu}^{cb} + \Gamma^\lambda_{\mu\nu} \delta {\omega_\lambda}^{ab} - \Gamma^\lambda_{\nu\mu} \delta {\omega_\lambda}^{ab} = \\
&= (\partial_\mu \delta {\omega_\nu}^{ab} + \Gamma^\lambda_{\mu\nu} \delta {\omega_\lambda}^{ab} + {\omega_\mu}^a{}_c\, \delta {\omega_\nu}^{cb} + {\omega_\mu}^b{}_c\, \delta {\omega_\nu}^{ac}) \\
&\quad - (\partial_\nu \delta {\omega_\mu}^{ab} + \Gamma^\lambda_{\nu\mu} \delta {\omega_\lambda}^{ab} + {\omega_\nu}^a{}_c\, \delta {\omega_\mu}^{cb} + {\omega_\nu}^b{}_c\, \delta {\omega_\mu}^{ac}) = \\
&= 2(\partial_\mu \delta {\omega_\nu}^{ab} + \Gamma^\lambda_{\mu\nu} \delta {\omega_\lambda}^{ab} + {\omega_\mu}^a{}_c\, \delta {\omega_\nu}^{cb} + {\omega_\mu}^b{}_c\, \delta {\omega_\nu}^{ac}) \equiv 2\nabla_\mu \delta {\omega_\nu}^{ab}
\end{aligned} \tag{2.113}
$$

where in the third equality we used the fact that the variation $\delta$ commutate with the partial derivative and we added a zero term at the end of the expression since the torsion free condition (2.106) of Riemann spacetime implies $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, in the fourth equality we used the antisymmetry of ${\omega_\nu}^{ab}$ given in eq. (2.112) and $\delta\eta_{ab} = 0$ to rewrite the following two terms

$$
\delta {\omega_\mu}^a{}_c\, {\omega_\nu}^{cb} = -{\omega_\nu}^{bc} \delta {\omega_\mu}^a{}_c = -{\omega_\nu}^b{}_c\, \delta {\omega_\mu}^{ac} \,, \tag{2.114}
$$

$$
-\delta {\omega_\nu}^a{}_c\, {\omega_\mu}^{cb} = {\omega_\mu}^{bc} \delta {\omega_\nu}^a{}_c = {\omega_\mu}^b{}_c\, \delta {\omega_\nu}^{ac} \,, \tag{2.115}
$$

in the fifth equality we used again the antisymmetry of ${\omega_\nu}^{ab}$ and in the last one we identify the terms in brackets to be $\nabla_\mu \delta {\omega_\nu}^{ab}$.

### 2.4.3 The tetrad Einstein-Hilbert action

In this section we want to show that the Einstein Hilbert action in tetrad formalism takes the following form

$$
S_{EH}^{tetrad} = \frac{1}{16\pi} \int_{\mathcal{M}} \frac{1}{2} \varepsilon_{abcd}\, e^a \wedge e^b \wedge R^{cd} \,, \tag{2.116}
$$

where $\varepsilon_{abcd}$ is the Levi-Civita symbol. Note that, as suggested by I. Jubb [14], we do not regard $S_{EH}^{tetrad}$ as a first order Palatini action because $R^{ab} \equiv R^{ab}(\omega^{ab}(e^a))$ is a function of $e^a$ and is determined by the condition that general relativity is a torsion free theory, i.e. $T^a = de^a + {\omega^a}_b \wedge e^b = 0$ obtained combining eq. (2.103) and eq. (2.106).

The first thing we want to do, is to compute the Ricci scalar

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} {R^\lambda}_{\mu\lambda\nu} = g^{\mu\nu} g^{\lambda\kappa} R_{\kappa\mu\lambda\nu} = {e^\mu}_a {e^\nu}_b \eta^{ab} {e^\lambda}_c {e^\kappa}_d \eta^{cd} R_{\kappa\mu\lambda\nu} = \\
&= {e^\kappa}_d {e^\mu}_a e^{\lambda d} e^{\nu a} R_{\kappa\mu\lambda\nu} = {e^\kappa}_d {e^\mu}_a {R_{\kappa\mu}}^{da} \,,
\end{aligned} \tag{2.117}
$$

where we used eq. (2.89) to express the metric in function of the tetrads and eq. (2.110) in the last equality. Defining $e := \sqrt{-g}$, we have

$$
S_{EH} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x\, \sqrt{-g} R = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x\, e\, {e^\mu}_a {e^\nu}_b {R_{\mu\nu}}^{ab} \,. \tag{2.118}
$$

Now, we show that eq. (2.116) is equivalent to eq. (2.118) proceeding as follows

$$
\begin{aligned}
\frac{1}{2}\varepsilon_{abcd}\, e^a \wedge e^b \wedge R^{cd} &= \frac{1}{2}\varepsilon_{abcd}\left(e_\mu{}^a dx^\mu\right)\wedge\left(e_\nu{}^b dx^\nu\right)\wedge\left(\frac{1}{2}R_{\kappa\lambda}{}^{cd}dx^\kappa\wedge dx^\lambda\right) = \\
&= \frac{1}{4}\varepsilon_{abcd}\, e_\mu{}^a e_\nu{}^b R_{\kappa\lambda}{}^{cd} dx^\mu\wedge dx^\nu\wedge dx^\kappa\wedge dx^\lambda = \\
&= \frac{1}{4}\varepsilon_{abcd}\, e_\mu{}^a e_\nu{}^b R_{\kappa\lambda}{}^{cd}\varepsilon^{\mu\nu\kappa\lambda} d^4x = \\
&= \frac{1}{4}e\,\varepsilon_{abcd}\varepsilon^{a'b'c'd'}\; e^\mu{}_{a'}e^\nu{}_{b'}e^\kappa{}_{c'}e^\lambda{}_{d'}e_\mu{}^a e_\nu{}^b R_{\kappa\lambda}{}^{cd}d^4x = \\
&= \frac{1}{4}e\,\varepsilon_{abcd}\varepsilon^{a'b'c'd'}\;\delta^a_{a'}\delta^b_{b'}e^\kappa{}_{c'}e^\lambda{}_{d'}R_{\kappa\lambda}{}^{cd}d^4x = \\
&= \frac{1}{2}e\left(\delta^{c'}_c\delta^{d'}_d - \delta^{c'}_d\delta^{d'}_c\right)e^\kappa{}_{c'}e^\lambda{}_{d'}R_{\kappa\lambda}{}^{cd}d^4x = \\
&= \frac{1}{2}e\left(e^\kappa{}_c e^\lambda{}_d - e^\kappa{}_d e^\lambda{}_c\right)R_{\kappa\lambda}{}^{cd}d^4x = \\
&= e\, e^\kappa{}_c e^\lambda{}_d R_{\kappa\lambda}{}^{cd}d^4x\,,
\end{aligned}
\tag{2.119}
$$

where in the fourth equality we used $\varepsilon^{\mu\nu\kappa\lambda} = e\varepsilon^{a'b'c'd'}e^\mu{}_{a'}e^\nu{}_{b'}e^\kappa{}_{c'}e^\lambda{}_{d'}$, in the sixth equality

$$
\varepsilon_{abcd}\varepsilon^{a'b'c'd'}\;\delta^a_{a'}\delta^b_{b'} = \varepsilon_{abcd}\varepsilon^{abc'd'} = 2!\,\delta^{c'd'}_{[cd]} = 2\det\begin{bmatrix}\delta^{c'}_c & \delta^{c'}_d\\ \delta^{d'}_c & \delta^{d'}_d\end{bmatrix} = 2\left(\delta^{c'}_c\delta^{d'}_d - \delta^{c'}_d\delta^{d'}_c\right)\,, \tag{2.120}
$$

and in the last one the antisymmetry of $R_{\mu\nu}{}^{ab}$ given in eq. (2.111).

We have therefore shown that eq. (2.116) is equivalent to eq. (2.118). Furthermore, note that we can use the result of eq. (2.119) to change from the compact tetrad notation to the coordinate basis notation

$$
S^{tetrad}_{EH} = \frac{1}{16\pi}\int_{\mathcal{M}}\frac{1}{2}\varepsilon_{abcd}\, e^a\wedge e^b\wedge R^{cd} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\, e\, e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^{ab}\,. \tag{2.121}
$$

To complete our picture of the tetrad Einstein-Hilbert action we should remark that few authors have pointed out, see for example C. Rovelli and F. Vidotto [16], that this action is not exactly equivalent to the metric one, but it differs by a sign factor. This sign factor play an important role in quantum field theory.

### 2.4.4 The boundary terms of the tetrad Einstein-Hilbert action

In the previous section we showed that using the tetrad formalism the Einstein-Hilbert action takes the following form

$$
S^{tetrad}_{EH} = \frac{1}{16\pi}\int_{\mathcal{M}}\frac{1}{2}\varepsilon_{abcd}\, e^a\wedge e^b\wedge R^{cd}\,, \tag{2.122}
$$

as you can guess, the advantage of this notation is the absence of the metric which will simplify the variation of the action. We will go through the derivation as first proposed by Jubb *et al.* [14], we will adapt their notation to our own notation.

We compute the variation of the action as follows

$$
\begin{aligned}
16\pi\, \delta S_{EH}^{tetrad} &= \int_{\mathcal{M}} \frac{1}{2}\varepsilon_{abcd}\left(\delta e^a \wedge e^b \wedge R^{cd} + e^a \wedge \delta e^b \wedge R^{cd} + e^a \wedge e^b \wedge \delta R^{cd}\right) = \\
&= \int_{\mathcal{M}} \varepsilon_{abcd}\left(\delta e^a \wedge e^b \wedge R^{cd} + e^a \wedge e^b \wedge \nabla \delta\omega^{cd}\right) \,, \qquad (2.123)
\end{aligned}
$$

where for the third term we used eq. (2.113) and for the second one we performed the following manipulation

$$
\varepsilon_{abcd}\, e^a \wedge \delta e^b \wedge R^{cd} = -\varepsilon_{abcd}\, \delta e^b \wedge e^a \wedge R^{cd} = \varepsilon_{abcd}\, \delta e^b \wedge e^a \wedge R^{dc} = \varepsilon_{badc}\, \delta e^b \wedge e^a \wedge R^{dc} \,, \quad (2.124)
$$

where we used the antisymmetry of $R_{\mu\nu}{}^{ab}$ given in eq. (2.111) and the antisymmetry of the Levi-Civita symbol $\varepsilon_{abcd}$. The first term in eq. (2.123) gives the Einstein field equations in vacuum, i.e. $\varepsilon_{abcd}\, e^b \wedge R^{cd} = 0$, while the second term is the boundary term. Using the tetrad postulate of eq. (2.94), the variation of the boundary term can be rewritten as

$$
\begin{aligned}
-16\pi\, \delta S_B \equiv \int_{\mathcal{M}} \varepsilon_{abcd}\, e^a \wedge e^b \wedge \nabla\delta\omega^{cd} &= \int_{\mathcal{M}} \nabla\left(\varepsilon_{abcd}\, e^a \wedge e^b \wedge \delta\omega^{cd}\right) = \\
= \int_{\partial\mathcal{M}} \varepsilon_{abcd}\, e^a \wedge e^b \wedge \delta\omega^{cd} &= \delta\left(\int_{\partial\mathcal{M}} \varepsilon_{abcd}\, e^a \wedge e^b \wedge \omega^{cd}\right) \,, \qquad (2.125)
\end{aligned}
$$

where in the third equality we used Gauss' theorem and in the last one, we took the the variation $\delta$ outside the integral demanding that the pullback of the metric to the boundary $\partial\mathcal{M}$ is unvaried and that the pullback of $e^a$ to the boundary $\partial\mathcal{M}$ has zero variation. It is now possible to read out the boundary term from eq. (2.125).

Up until to this point our derivation is independent of the type of boundary $\partial\mathcal{M}$. We want to show that we can recover all types of boundary terms. To do that, we express the boundary term in the coordinate basis using a similar procedure as the one used to obtain eq. (2.121)

$$
\begin{aligned}
S_B = -\frac{1}{16\pi}\int_{\partial\mathcal{M}} \varepsilon_{abcd}\, e^a \wedge e^b \wedge \omega^{cd} &= -\frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3y\, 2e\, e_\mu{}^{\hat{a}} e_{\hat{a}}\, e^\mu{}_a e^\nu{}_b \omega_\nu{}^{ab} = \\
= -\frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3y\, 2e\, e_{\hat{a}}\, e^\nu{}_b \omega_\nu{}^{\hat{a}b} &= \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3y\, 2e\, e_{\hat{a}}\left(g^{\mu\nu} - e^\mu{}_{\hat{a}} e^{\nu\hat{a}}\right) D_\mu e_\nu{}^{\hat{a}} \,, \qquad (2.126)
\end{aligned}
$$

where in the second equality we have introduced $e_\mu{}^{\hat{a}} e_{\hat{a}}$, which is the normal to the surface expressed in the tetrad basis and we used the index $\hat{a}$ to indicate the type of boundary, in the third equality we used $e_\mu{}^{\hat{a}} e^\mu{}_a = \delta_a^{\hat{a}}$ and in the last one the following manipulation

$$
e^\mu{}_b \omega_\mu{}^{\hat{a}b} = -e^\mu{}_b e_\kappa{}^b D_\mu e^{\kappa\hat{a}} = -\left(\delta^\mu_\kappa - e^\mu{}_{\hat{a}} e_\kappa{}^{\hat{a}}\right) D_\mu (e_\nu{}^{\hat{a}} g^{\nu\kappa}) = -\left(g^{\mu\nu} - e^\mu{}_{\hat{a}} e^{\nu\hat{a}}\right) D_\mu e_\nu{}^{\hat{a}} \,, \quad (2.127)
$$

where in the first equality we used the definition of $\omega_\nu{}^{\hat{a}b}$ given in eq. (2.93), in the second equality the fact that the sum over $b$ extends over all indices except $\hat{a}$ because of the antisymmetry of $\omega_\nu{}^{\hat{a}b}$ and in the last one $D_\lambda g^{\mu\nu} \equiv \nabla_\lambda g^{\mu\nu} = 0$.

Let us first consider non-null surfaces. For this case the normal to the surface is $n_\mu = e_\mu{}^{\hat{a}} e_{\hat{a}}$ as defined in eq. (2.6). Then, $D_\mu e_\nu{}^{\hat{a}} = e^{\hat{a}} \nabla_\mu n_\nu$ and with $e^{\mu\hat{a}} = g^{\mu\nu} e_\nu{}^{\hat{a}} = g^{\mu\nu} n_\nu e^{\hat{a}} = n^\mu e^{\hat{a}}$, we calculate

$$
e^\mu{}_{\hat{a}} e^{\nu\hat{a}} = \eta_{\hat{a}\hat{b}} e^{\mu\hat{b}} e^{\nu\hat{a}} = \eta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}} n^\mu n^\nu = \varepsilon n^\mu n^\nu \,, \qquad (2.128)
$$

where $\varepsilon \equiv \eta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}} = \eta_{\hat{a}\hat{b}} e_\mu{}^{\hat{a}} e_\nu{}^{\hat{b}} n^\mu n^\nu = g_{\mu\nu} n^\mu n^\nu$ is given in eq. (2.7). Thus, we recover

$$e_{\hat{a}} \left(g^{\mu\nu} - e^\mu{}_{\hat{a}} e^{\nu\hat{a}}\right) D_\mu e_\nu{}^{\hat{a}} = e_{\hat{a}} e^{\hat{a}} \left(g^{\mu\nu} - \varepsilon n^\mu n^\nu\right) \nabla_\mu n_\nu = \varepsilon h^{\mu\nu} \nabla_\mu n_\nu = \varepsilon K \,, \tag{2.129}$$

where we used again $e_{\hat{a}} e^{\hat{a}} = \eta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}} \equiv \varepsilon$, the definition of the induced metric $h^{\mu\nu}$ given in eq. (2.9) and the trace of the extrinsic curvature defined in eq. (2.30). We see that this is exactly the Gibbons-Hawking-York boundary term of eq. (2.33), substituting $e = \sqrt{|h|}$ and eq. (2.129) in eq. (2.127) we get

$$S_B = \frac{1}{16\pi} \int_{\partial\mathcal{M}} d^3y \, \varepsilon \sqrt{|h|} 2K \equiv S_{GHY} \,. \tag{2.130}$$

Finally, we consider null surfaces. For this case we use the indices $\hat{a} = 0, 1$ to differentiate the normal to the surface $l_\mu = e_\mu{}^0 e_0$ to the auxiliary vector $k_\nu = e_\nu{}^1 e_1$ in the tetrad basis. Furthermore we demand that everywhere holds $-1 = l_\mu k^\mu = g_{\mu\nu} e^\mu{}_0 e^\nu{}_1 e^0 e^1 = \eta_{01} e^0 e^1$, where in the last equality we used eq. (2.89). Then, $D_\mu e_\nu{}^0 = e^0 \nabla_\mu l_\nu$ and with $e^{\mu\hat{a}} = g^{\mu\nu} e_\nu{}^{\hat{a}}$, we compute

$$e^\mu{}_{\hat{a}} e^{\nu\hat{a}} = \eta_{\hat{a}\hat{b}} e^{\mu\hat{b}} e^{\nu\hat{a}} = \eta_{01} e^{\mu 1} e^{\nu 0} = \eta_{01} e^0 e^1 k^\mu l^\nu = -k^\mu l^\nu \,. \tag{2.131}$$

Thus, we recover

$$e_{\hat{a}} \left(g^{\mu\nu} - e^\mu{}_{\hat{a}} e^{\nu\hat{a}}\right) D_\mu e_\nu{}^{\hat{a}} = e_0 e^0 \left(g^{\mu\nu} + k^\mu l^\nu\right) \nabla_\mu l_\nu = \Pi^{\mu\nu} \nabla_\mu l_\nu = \Theta + \kappa \,, \tag{2.132}$$

where we used the identity $e_0 e^0 = \eta_{00} e^0 e^0 = 1$, eq. (2.46) to get $\Pi^{\mu\nu}$ and eq. (2.59) to obtain the final result. We see that this is exactly the null boundary term of eq. (2.81) for $A = 1$, substituting $d^3y \, e = d\lambda dz^2 \sqrt{q}$ and eq. (2.132) in eq. (2.127) we get

$$S_B = \frac{1}{16\pi} \int_{\partial\mathcal{M}} d\lambda d^2z \, 2\sqrt{q}(\Theta + \kappa) \equiv S_{null} \,. \tag{2.133}$$

We have therefore showed that starting from the tetrad Einstein-Hilbert action is possible to derive any type of boundary term. The tetrads procedure is more compact and elegant in the sense that we do not have to carry out the variation of the metric.

# 3 General spacetimes

## 3.1 Overview

In the previous chapters we saw that the Einstein-Hilbert action of Riemann spacetime must be supplied with an extra term, i.e. $S_{GHY}$ or $S_{null}$, in order to recover the Einstein field equations from the variational principle when boundaries of the spacetime manifold are considered. Here, we are going to explore different types of spacetimes geometries and compare their actions with the one of conventional general relativity.

First of all, we give an overview of the different types of metric affine geometries using the book of T. Ortin [17] adapting it to our own notation.

Second, we study the case of teleparallel general relativity which is based on the concept of distant parallelism, first proposed by Einstein when he introduced the tetrad field trying to unify general relativity and electromagnetism [18]. The modern notion of parallelism, for two vectors separated by a finite distance in spacetime, was rigorously defined by Weitzenböck when he introduced the Weitzenböck spacetime for which the Riemann tensor vanishes.

At last, we explore the case of symmetric teleparallel general relativity which has raised attention in many recent publications. We are interested in these alternative descriptions of general relativity because, as we will show, their gravitational actions already incorporate the boundary counter-term usually added by hand to the Einstein-Hilbert action.

## 3.2 Metric affine geometries

Let $\mathcal{M}(\widetilde{\Gamma}, g)$ be a manifold with an arbitrary affine connection $\widetilde{\Gamma}^{\alpha}{}_{\mu\nu}$ and metric $g_{\mu\nu}$. We define the curvature tensor $R^{\alpha}{}_{\mu\beta\nu}$ and the torsion tensor $T^{\alpha}{}_{\mu\nu}$ through the Ricci identities for a scalar $\phi$, a vector $\xi^{\alpha}$ and a one-form $\omega_{\alpha}$:

$$[\widetilde{\nabla}_{\mu}, \widetilde{\nabla}_{\nu}]\, \phi = -T^{\alpha}{}_{\mu\nu} \widetilde{\nabla}_{\alpha} \phi \,, \tag{3.1}$$

$$[\widetilde{\nabla}_{\mu}, \widetilde{\nabla}_{\nu}]\, \xi^{\alpha} = R^{\alpha}{}_{\beta\mu\nu} \xi^{\beta} - T^{\alpha}{}_{\mu\nu} \widetilde{\nabla}_{\beta} \xi^{\alpha} \,, \tag{3.2}$$

$$[\widetilde{\nabla}_{\mu}, \widetilde{\nabla}_{\nu}]\, \omega_{\beta} = -R^{\alpha}{}_{\beta\mu\nu} \omega_{\alpha} - T^{\alpha}{}_{\mu\nu} \widetilde{\nabla}_{\alpha} \omega_{\beta} \,, \tag{3.3}$$

where the covariant derivative $\widetilde{\nabla}_{\mu}$ is the one compatible with the affine connection $\widetilde{\Gamma}^{\alpha}{}_{\mu\nu}$. Furthermore, the Riemann tensor $R^{\alpha}{}_{\mu\beta\nu}$ describing the curvature of spacetime is given by the following expression[7]

$$R^{\alpha}{}_{\mu\beta\nu} = \widetilde{\Gamma}^{\alpha}_{\mu\nu,\beta} - \widetilde{\Gamma}^{\alpha}_{\mu\beta,\nu} + \widetilde{\Gamma}^{\alpha}_{\lambda\beta} \widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\lambda\nu} \widetilde{\Gamma}^{\lambda}_{\mu\beta} \,, \tag{3.4}$$

which satisfy the antisymmetry conditions $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu}$ and the torsion tensor by

$$T^{\alpha}{}_{\mu\nu} = \widetilde{\Gamma}^{\alpha}{}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}{}_{\nu\mu} \tag{3.5}$$

which satisfy the antisymmetry condition $T^{\alpha}{}_{\mu\nu} = -T^{\alpha}{}_{\nu\mu}$.

The most general affine connection one could define for an affinely connected metric spacetime is

$$\widetilde{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu} \,, \tag{3.6}$$

[7] We keep the convention stated in eq. (1.2) which may differs from the one used in recent literature.

where $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols computed from the metric $g_{\mu\nu}$ as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left( g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right) , \tag{3.7}$$

$K^{\alpha}{}_{\mu\nu}$ is known as the *contorsion tensor* and it is computed using the torsion tensor $T^{\alpha}{}_{\mu\nu}$ as

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left( T_{\mu}{}^{\alpha}{}_{\nu} + T_{\nu}{}^{\alpha}{}_{\mu} - T^{\alpha}{}_{\mu\nu} \right) , \tag{3.8}$$

which satisfy the antisymmetry condition $K_{\alpha\mu\nu} = \frac{1}{2}(T_{\mu\alpha\nu} + T_{\nu\alpha\mu} - T_{\alpha\mu\nu}) = -\frac{1}{2}(-T_{\mu\alpha\nu} + T_{\nu\mu\alpha} + T_{\alpha\mu\nu}) = -K_{\mu\alpha\nu}$, and $L^{\alpha}{}_{\mu\nu}$ is known as the *disformation tensor* and it is computed using the non-metricity $Q_{\alpha\mu\nu} = \widetilde{\nabla}_{\alpha} g_{\mu\nu}$, where the covariant derivative $\widetilde{\nabla}_{\alpha}$ is the one compatible with the affine connection $\widetilde{\Gamma}^{\alpha}{}_{\mu\nu}$, as

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2}(Q^{\alpha}{}_{\mu\nu} - Q_{\mu}{}^{\alpha}{}_{\nu} - Q_{\nu}{}^{\alpha}{}_{\mu}) . \tag{3.9}$$

In Figure 3.1 we present a diagram showing different types of affinely connected metric spacetimes, which summaries what we are going to explain in the following two paragraphs.

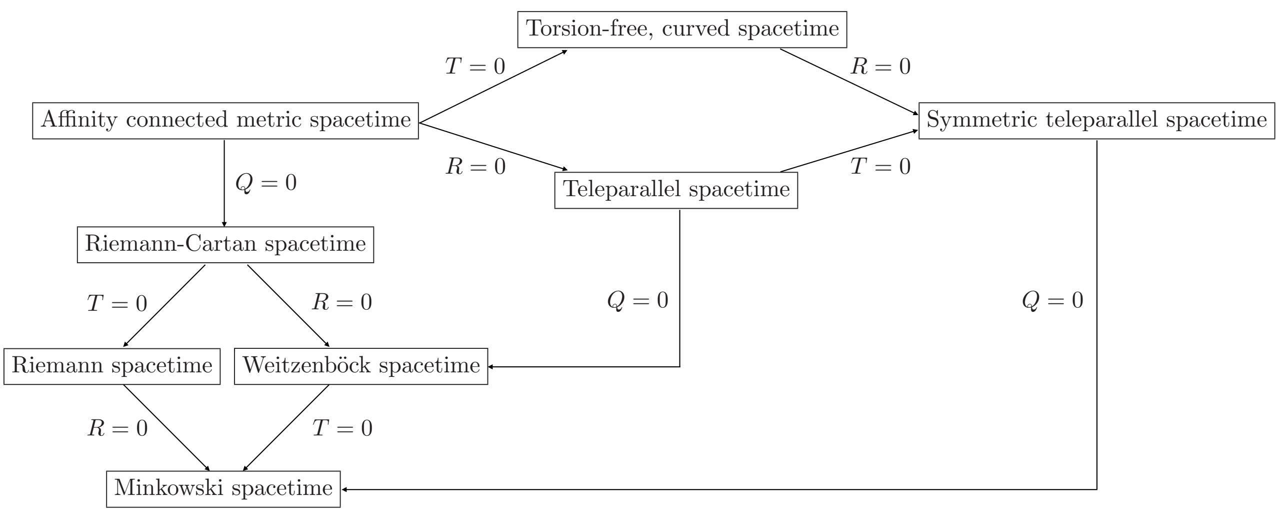


Figure 3.1: Diagram of affinely connected metric spacetimes showing the different permutations of the non-metricity tensor $Q^{\alpha}{}_{\mu\nu}$, the Riemann curvature tensor $R^{\alpha}{}_{\mu\beta\nu}$ and the torsion tensor $T^{\alpha}{}_{\mu\nu}$ where the indices were suppressed to keep the notation compact.

The Riemann-Cartan spacetime is obtained when the non-metricity tensor vanishes, i.e. $Q_{\alpha\mu\nu} = 0$, this is known as the *metric postulate*. The metric postulate leaves the torsion undetermined. In order to have a connection completely determined by the metric one has to impose the vanishing of the torsion, i.e. $T^{\alpha}{}_{\mu\nu} = 0 \Rightarrow \widetilde{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu}$. A Riemann-Cartan spacetime with vanishing torsion is a Riemann spacetime. Another way to determinate the connection is to impose the vanishing of the curvature tensor. In this case the connection is called Weitzenböck connection and, as we are going to see, is determined by the tetrads. A Riemann-Cartan spacetime with Weitzenböck connection is a Weitzenböck spacetime.

Furthermore, one should note that by setting the curvature of Riemann spacetime or the torsion of Weitzenböck spacetime to zero, one recovers Minkowski spacetime.

Other spacetimes are found when we consider a non vanishing non-metricity $Q_{\alpha\mu\nu}$. For vanishing curvature one recovers the teleparallel spacetime which become equivalent to the Weitzenböck spacetime when the non-metricity vanishes. Another interesting spacetime, which we are going to study, is the symmetric teleparallel spacetime where both the curvature and torsion vanish.

## 3.3 Teleparallel general relativity

### 3.3.1 Teleparallel action in Weitzenböck spacetime

Before proceeding, we spend a few words on the notation. To avoid any confusion, we will explicitly specify the dependence on the Weitzenböck connection for objects of Weitzenböck spacetime, as for example with the curvature of the Weitzenböck spacetime $\widetilde{R}^{\alpha}{}_{\mu\beta\nu}(\widetilde{\Gamma}) = 0$.

To describe the Weitzenböck Spacetime we use the already introduced tetrad formalism. The tetrad postulate given in eq. (2.94) takes the following form in Weitzenböck spacetime

$$\widetilde{\nabla}_\mu e_\nu{}^a \equiv \partial_\mu e_\nu{}^a - e_\kappa{}^a \widetilde{\Gamma}^\kappa{}_{\mu\nu} + \widetilde{\omega}_\mu{}^a{}_b e_\nu{}^b = 0\,, \tag{3.10}$$

where $\widetilde{\nabla}_\mu$ is the covariant derivative compatible with the Weitzenböck connection. We can use the trivial solution $\widetilde{\omega}_\mu{}^a{}_b = 0$ of eq. (2.110) to determinate the Weitzenböck connection, this result is found as follows

$$\widetilde{R}_{\mu\nu}{}^{ab}(\widetilde{\omega}) = e^{\lambda a} e^{\kappa b} \widetilde{R}_{\mu\nu\lambda\kappa}(\widetilde{\Gamma}) = 0 \;\Rightarrow\; \widetilde{\omega} = 0\,, \tag{3.11}$$

where we used that the Riemann curvature of Weitzenböck spacetime vanishes.

Plugging the result of eq. (3.11) in eq. (3.10) and using the tetrad identity (2.88), we get the following expression for the Weitzenböck connection

$$\widetilde{\Gamma}^\kappa{}_{\mu\nu} = e^\kappa{}_a \partial_\mu e_\nu{}^a\,. \tag{3.12}$$

Now that we have the connection, we can explicitly write the torsion tensor as

$$T^\lambda{}_{\mu\nu} = \widetilde{\Gamma}^\lambda{}_{\mu\nu} - \widetilde{\Gamma}^\lambda{}_{\nu\mu} = e^\lambda{}_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)\,. \tag{3.13}$$

Since in Weitzenböck spacetime the non-metricity tensor vanishes, according to eq. (3.6) the contorsion tensor is the difference between the Weitzenböck connection and the Levi-Civita connection

$$K^\alpha{}_{\mu\nu} = \widetilde{\Gamma}^\alpha{}_{\mu\nu} - \Gamma^\alpha_{\mu\nu}\,. \tag{3.14}$$

Now that we have expressed the torsion of the Weitzenböck spacetime in function of the tetrad fields, we define the teleparallel action as

$$S_T := \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\, e\, \mathbb{T} \tag{3.15}$$

where $e = \sqrt{-g}$ is the determinant of the tetrad $e^\mu{}_a$ and $\mathbb{T}$ is the torsion scalar defined as

$$\mathbb{T} \equiv -\frac{1}{4} T^\lambda{}_{\mu\nu} T_\lambda{}^{\mu\nu} + \frac{1}{2} T^{\mu\nu}{}_\lambda T^\lambda{}_{\mu\nu} + T^\lambda{}_{\lambda\alpha} T^\sigma{}_\sigma{}^\alpha\,. \tag{3.16}$$

### 3.3.2 The torsion scalar

In order to show that the variation of the teleparallel action $S_T$ gives exactly the Einstein field equations without the need of adding a boundary counter-term, we need to relate the torsion scalar $\mathbb{T}$ of Weitzenböck spacetime to the Ricci scalar of Riemann spacetime. To do that we use the fact that the Riemann tensor of Weitzenböck spacetime vanishes.

Before proceeding with the calculations we point out that we are following a different notation that the one usually found in the recent literature. Our convention of the Riemann tensor and contortion tensor agree for example with the one used by B. Li *et al.* [19]. One may verify our results with the one presented in this paper, note that our convention for the torsion scalar $\mathbb{T}$ differs from their convention by a sign factor.

We proceed as follows

$$\begin{aligned}
0 = \widetilde{R}^{\alpha}{}_{\mu\beta\nu}(\widetilde{\Gamma}) &= \widetilde{\Gamma}^{\alpha}_{\mu\nu,\beta} - \widetilde{\Gamma}^{\alpha}_{\mu\beta,\nu} + \widetilde{\Gamma}^{\alpha}_{\lambda\beta}\widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\lambda\nu}\widetilde{\Gamma}^{\lambda}_{\mu\beta} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + K^{\alpha}{}_{\mu\nu,\beta} - K^{\alpha}{}_{\mu\beta,\nu} + K^{\alpha}{}_{\lambda\beta}K^{\lambda}{}_{\mu\nu} - K^{\alpha}{}_{\lambda\nu}K^{\lambda}{}_{\mu\beta} \\
&\qquad + \Gamma^{\alpha}_{\lambda\beta}K^{\lambda}{}_{\mu\nu} - \Gamma^{\alpha}_{\lambda\nu}K^{\lambda}{}_{\mu\beta} + \Gamma^{\lambda}_{\mu\nu}K^{\alpha}{}_{\lambda\beta} - \Gamma^{\lambda}_{\mu\beta}K^{\alpha}{}_{\lambda\nu} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + (\partial_\beta K^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}_{\lambda\beta}K^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}_{\mu\beta}K^{\alpha}{}_{\lambda\nu} - \Gamma^{\lambda}_{\nu\beta}K^{\alpha}{}_{\mu\lambda}) \\
&\qquad - (\partial_\nu K^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}_{\lambda\nu}K^{\lambda}{}_{\mu\beta} - \Gamma^{\lambda}_{\mu\nu}K^{\alpha}{}_{\lambda\beta} - \Gamma^{\lambda}_{\beta\nu}K^{\alpha}{}_{\mu\lambda}) \\
&\qquad + K^{\alpha}{}_{\lambda\beta}K^{\lambda}{}_{\mu\nu} - K^{\alpha}{}_{\lambda\nu}K^{\lambda}{}_{\mu\beta} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + \nabla_\beta K^{\alpha}{}_{\mu\nu} - \nabla_\nu K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\lambda\beta}K^{\lambda}{}_{\mu\nu} - K^{\alpha}{}_{\lambda\nu}K^{\lambda}{}_{\mu\beta}\,,
\end{aligned} \tag{3.17}$$

where in the second equality we used the definition of contorsion given in eq. (3.14) to rewrite the Weitzenböck connection, in the third equality we add a zero term $\Gamma^{\lambda}_{\beta\nu}K^{\alpha}{}_{\mu\lambda} - \Gamma^{\lambda}_{\nu\beta}K^{\alpha}{}_{\mu\lambda} = 0$ since the Levi-Civita connection of the conventional general relativity is symmetric $\Gamma^{\lambda}_{\beta\nu} = \Gamma^{\lambda}_{\nu\beta}$ and in the last equality we identified the covariant derivative of the contorsion compatible with the Levi-Civita connection, i.e. $\nabla_\beta K^{\alpha}{}_{\mu\nu}$. Rearranging the terms of eq. (3.17) we find

$$R^{\alpha}{}_{\mu\beta\nu} \equiv R^{\alpha}{}_{\mu\beta\nu}(\Gamma) = \nabla_\nu K^{\alpha}{}_{\mu\beta} - \nabla_\beta K^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\lambda\nu}K^{\lambda}{}_{\mu\beta} - K^{\alpha}{}_{\lambda\beta}K^{\lambda}{}_{\mu\nu}\,, \tag{3.18}$$

which we use to compute the Ricci scalar. Taking the trace of the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, we find

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu} = \nabla_\mu K^{\alpha\mu}{}_{\alpha} - \nabla_\alpha K^{\alpha\mu}{}_{\mu} + K^{\alpha}{}_{\sigma\nu}K^{\sigma\nu}{}_{\alpha} - K^{\alpha}{}_{\sigma\alpha}K^{\sigma\nu}{}_{\nu}\,. \tag{3.19}$$

The first two terms in eq. (3.19) can be rewritten using the definition of the contorsion given in eq. (3.8) as

$$\begin{aligned}
\nabla_\mu K^{\alpha\mu}{}_{\alpha} - \nabla_\alpha K^{\alpha\mu}{}_{\mu} &= \frac{1}{2}\nabla_\mu (T^{\mu\alpha}{}_{\alpha} + T_{\alpha}{}^{\alpha\mu} - T^{\alpha\mu}{}_{\alpha}) - \frac{1}{2}\nabla_\alpha (T^{\mu\alpha}{}_{\mu} + T_{\mu}{}^{\alpha\mu} - T^{\alpha\mu}{}_{\mu}) = \\
&= -\nabla_\mu T^{\alpha\mu}{}_{\alpha} - \nabla_\mu T_{\alpha}{}^{\mu\alpha} + \nabla_\mu T^{\mu\alpha}{}_{\alpha} = \\
&= -2\nabla_\mu T^{\alpha\mu}{}_{\alpha}\,,
\end{aligned} \tag{3.20}$$

where in the second equality we used $T_{\alpha}{}^{\alpha\mu} = -T_{\alpha}{}^{\mu\alpha}$ to combine the second terms in the two brackets and in the last equality $T_{\alpha}{}^{\mu\alpha} = T^{\alpha\mu}{}_{\alpha}$ together with $T^{\mu\alpha}{}_{\alpha} = 0$ easily verifiable with

the definition of the torsion given in eq. (3.13).
The third term in eq. (3.19) is rewritten as follows

$$\begin{aligned} K^{\alpha}{}_{\sigma\nu}K^{\sigma\nu}{}_{\alpha} &= \frac{1}{4}(T_{\nu}{}^{\alpha}{}_{\sigma} + T_{\sigma}{}^{\alpha}{}_{\nu} - T^{\alpha}{}_{\sigma\nu})(T_{\alpha}{}^{\sigma\nu} + T^{\nu\sigma}{}_{\alpha} - T^{\sigma\nu}{}_{\alpha}) = \\ &= \frac{1}{4}(T_{\nu}{}^{\alpha}{}_{\sigma}T_{\alpha}{}^{\sigma\nu} + T_{\nu}{}^{\alpha}{}_{\sigma}T^{\nu\sigma}{}_{\alpha} - T_{\nu}{}^{\alpha}{}_{\sigma}T^{\sigma\nu}{}_{\alpha} + T_{\sigma}{}^{\alpha}{}_{\nu}T_{\alpha}{}^{\sigma\nu} + T_{\sigma}{}^{\alpha}{}_{\nu}T^{\nu\sigma}{}_{\alpha} - T_{\sigma}{}^{\alpha}{}_{\nu}T^{\sigma\nu}{}_{\alpha}) \\ &\quad - \frac{1}{4}T^{\alpha}{}_{\sigma\nu}T_{\alpha}{}^{\sigma\nu} - \frac{1}{4}T^{\alpha}{}_{\sigma\nu}T^{\nu\sigma}{}_{\alpha} + \frac{1}{4}T^{\alpha}{}_{\sigma\nu}T^{\sigma\nu}{}_{\alpha} = \\ &= -\frac{1}{4}T^{\alpha}{}_{\sigma\nu}T_{\alpha}{}^{\sigma\nu} + \frac{1}{2}T^{\sigma\nu}{}_{\alpha}T^{\alpha}{}_{\sigma\nu}\,, \end{aligned} \tag{3.21}$$

where in the last equality we used the antisymmetry propriety of $T^{\alpha}{}_{\mu\nu} = -T^{\alpha}{}_{\nu\mu}$ to rewrite the last three terms in the brackets as follows $T_{\sigma}{}^{\alpha}{}_{\nu}T_{\alpha}{}^{\sigma\nu} = -T_{\sigma}{}^{\alpha}{}_{\nu}T_{\alpha}{}^{\nu\sigma} = -T_{\nu}{}^{\alpha}{}_{\sigma}T_{\alpha}{}^{\sigma\nu}$; $T_{\sigma}{}^{\alpha}{}_{\nu}T^{\nu\sigma}{}_{\alpha} = T_{\sigma\nu}{}^{\alpha}T^{\nu}{}_{\alpha}{}^{\sigma} = T^{\sigma\nu}{}_{\alpha}T_{\nu}{}^{\alpha}{}_{\sigma}$; $-T_{\sigma}{}^{\alpha}{}_{\nu}T^{\sigma\nu}{}_{\alpha} = -T_{\nu}{}^{\alpha}{}_{\sigma}T^{\nu\sigma}{}_{\alpha}$ and we see that they all compensate the others in the brackets.
The fourth term in eq. (3.19) is rewritten as follows

$$-K^{\alpha}{}_{\sigma\alpha}K^{\sigma\nu}{}_{\nu} = -\frac{1}{4}(T_{\sigma}{}^{\alpha}{}_{\alpha} + T_{\alpha}{}^{\alpha}{}_{\sigma} - T^{\alpha}{}_{\sigma\alpha})(T^{\nu\sigma}{}_{\nu} + T_{\nu}{}^{\sigma\nu} - T^{\sigma\nu}{}_{\nu}) = T^{\alpha}{}_{\alpha\sigma}T^{\nu}{}_{\nu}{}^{\sigma}\,, \tag{3.22}$$

where we used $T_{\sigma}{}^{\alpha}{}_{\alpha} = T^{\sigma\nu}{}_{\nu} = 0$; $T_{\alpha}{}^{\alpha}{}_{\sigma} = T^{\alpha}{}_{\alpha\sigma}$ and the antisymmetry of the torsion tensor to rewrite $-T^{\alpha}{}_{\sigma\alpha} = T^{\alpha}{}_{\alpha\sigma}$, $T^{\nu\sigma}{}_{\nu} = -T^{\nu}{}_{\nu}{}^{\sigma}$ and $T_{\nu}{}^{\sigma\nu} = -T_{\nu}{}^{\nu\sigma} = -T^{\nu}{}_{\nu}{}^{\sigma}$.
Combining the third and fourth term of eq. (3.19), we obtain

$$K^{\alpha}{}_{\sigma\nu}K^{\sigma\nu}{}_{\alpha} - K^{\alpha}{}_{\sigma\alpha}K^{\sigma\nu}{}_{\nu} = -\frac{1}{4}T^{\alpha}{}_{\sigma\nu}T_{\alpha}{}^{\sigma\nu} + \frac{1}{2}T^{\sigma\nu}{}_{\alpha}T^{\alpha}{}_{\sigma\nu} + T^{\alpha}{}_{\alpha\sigma}T^{\nu}{}_{\nu}{}^{\sigma} \equiv \mathbb{T} \tag{3.23}$$

Plugging the results of eqs. (3.20) and (3.23) in eq. (3.19), we obtain

$$R = \mathbb{T} - 2\nabla_{\mu}T^{\alpha\mu}{}_{\alpha} \;\Leftrightarrow\; \mathbb{T} = R + 2\nabla_{\mu}T^{\alpha\mu}{}_{\alpha}\,, \tag{3.24}$$

this is a useful relation which relates the torsion scalar of the teleparallel theory with the Ricci scalar of the conventional general relativity.

### 3.3.3 The variational principle

We now show that the teleparallel action of general relativity constitutes a well-posed solution for the variational principle of a spacetime manifold with boundaries. We use the results of the previous section to rewrite the teleparallel action as

$$S_T = \frac{1}{16\pi}\int_M d^4x\, e\mathbb{T} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\, e(R + 2\nabla_{\mu}T^{\alpha\mu}{}_{\alpha}) \equiv S_{EH} + S_{torsion}\,. \tag{3.25}$$

We are going to show that the boundary counter-term that we usually add by hand to the Einstein-Hilbert action is already embedded in the teleparallel action. To do that, we follow the procedure presented in the paper of N. Oshita and Y. Wu [20].

The first thing we do is to express the metric tensor using the tetrad field as in eq. (2.90), i.e. $g_{\mu\nu} = \eta_{ab}e_{\mu}{}^{a}e_{\nu}{}^{b}$. We variate this expression and obtain $\delta g_{\mu\nu} = \eta_{ab}(\delta e_{\mu}{}^{a}e_{\nu}{}^{b} + e_{\mu}{}^{a}\delta e_{\nu}{}^{b})$.

If $g_{\mu\nu}$ is fixed, then this is equivalent to $\delta e_\mu{}^a e_\nu{}^b = -e_\mu{}^a \delta e_\nu{}^b$ for any $e_\mu{}^a$. In the variational problem we usually fix the metric $g_{\mu\nu}$ on the boundary of the manifold $\mathcal{M}$, therefore the boundary condition is equivalent to

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0 \;\Leftrightarrow\; \delta e_\mu{}^a|_{\partial\mathcal{M}} = 0\,. \tag{3.26}$$

Note that this condition fixes only ten components of $e^a_\mu$. The induced metric defined in eq. (2.9) can be rewritten using the tetrads as follows

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu = (\eta_{ab} - \varepsilon n_a n_b) e_\nu{}^a e_\mu{}^b\,, \tag{3.27}$$

where $n_a = n_\mu e^\mu{}_a$ is the component $a$ of the unit normal vector to the boundary expressed in the tetrad basis. Using the projector operator $h^\mu{}_\nu$ defined in eq. (2.10) we easily obtain the following decomposition of the tetrad field on the boundary

$$e^\mu{}_a = h^\mu{}_a + \varepsilon n^\mu n_a\,. \tag{3.28}$$

Note that since the tetrad field is fixed on the boundary, the derivative tangent to the surface must vanish, i.e.

$$h^\mu{}_a \partial_\mu \delta e_\nu{}^b = 0\,. \tag{3.29}$$

Starting from eq. (2.20), we compute the variation of the Einstein-Hilbert action using the tetrad formalism as follows

$$16\pi\,\delta S_{EH} = \int_{\mathcal{M}} d^4x\,\sqrt{-g} G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|} n_\mu \big(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\beta\alpha}\big) = \tag{3.30}$$

$$= \int_{\mathcal{M}} d^4x\,\sqrt{-g} G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|} n^\mu g^{\alpha\beta}\big(\partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta}\big) \tag{3.31}$$

where we used $\delta g^{\mu\lambda}|_{\partial\mathcal{M}} = \delta g^{\beta\lambda}|_{\partial\mathcal{M}} = 0$, to compute

$$\delta\Gamma^\mu_{\alpha\beta}|_{\partial\mathcal{M}} = \frac{1}{2} g^{\mu\lambda}\Big(\partial_\alpha(\delta g_{\lambda\beta}) + \partial_\beta(\delta g_{\alpha\lambda}) - \partial_\lambda(\delta g_{\alpha\beta})\Big)\,, \tag{3.32}$$

$$\delta\Gamma^\beta_{\beta\alpha}|_{\partial\mathcal{M}} = \frac{1}{2} g^{\beta\lambda}\Big(\partial_\beta(\delta g_{\lambda\alpha}) + \partial_\alpha(\delta g_{\beta\lambda}) - \partial_\lambda(\delta g_{\beta\alpha})\Big) = \frac{1}{2} g^{\beta\lambda}\partial_\alpha(\delta g_{\beta\lambda})\,, \tag{3.33}$$

and with $n_\mu = n^\nu g_{\mu\nu}$ we find

$$n_\mu(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\beta\alpha})|_{\partial\mathcal{M}} = n^\nu \frac{1}{2}\Big(g^{\alpha\beta}\delta^\lambda_\nu(\partial_\alpha(\delta g_{\lambda\beta}) + \partial_\beta(\delta g_{\alpha\lambda}) - \partial_\lambda(\delta g_{\alpha\beta})) - \delta^\alpha_\nu g^{\beta\lambda}\partial_\alpha(\delta g_{\beta\lambda})\Big) =$$

$$= n^\nu g^{\alpha\beta}(\partial_\alpha(\delta g_{\nu\beta}) - \partial_\nu(\delta g_{\alpha\beta}))\,. \tag{3.34}$$

Carrying on the manipulations of the term $\delta S_{EH}$ where we left them in eq. (3.31), we obtain the following for the first term in the surface integral

$$\partial_\alpha \delta g_{\mu\beta} = \eta_{ab}(e_\beta{}^b \partial_\alpha \delta e_\mu{}^a + \delta e_\mu{}^a \partial_\alpha e_\beta{}^b + e_\mu{}^a \partial_\alpha \delta e_\beta{}^b + \delta e_\beta{}^b \partial_\alpha e_\mu{}^a) =$$

$$= \eta_{ab}(e_\beta{}^b \partial_\alpha \delta e_\mu{}^a + e_\mu{}^a \partial_\alpha \delta e_\beta{}^b)\,, \tag{3.35}$$

because the variation of $e_\mu{}^a$ vanish on $\partial\mathcal{M}$. Furthermore, the first term of eq. (3.35) combines with the factor $n^\mu g^{\alpha\beta}$ of eq. (3.31) to form

$$n^\mu g^{\alpha\beta}\eta_{ab}e_\beta{}^b\partial_\alpha\delta e_\mu{}^a = n^\mu e^\alpha{}_a\partial_\alpha\delta e_\mu{}^a\,, \tag{3.36}$$

while second term of eq. (3.35) combines with the factor $n^\mu g^{\alpha\beta}$ of eq. (3.31) to form

$$\begin{aligned} n^\mu g^{\alpha\beta}\eta_{ab}e_\mu{}^a\partial_\alpha\delta e_\beta{}^b &= n^\mu(h^{\alpha\beta}+\varepsilon n^\alpha n^\beta)\eta_{ab}e_\mu{}^a\partial_\alpha\delta e_\beta{}^b = \\ &= n^\mu\eta_{ab}e_\mu{}^a e^{\beta c}h^\alpha{}_c\partial_\alpha\delta e_\beta{}^b + \varepsilon n^\alpha n^\beta n^\mu\eta_{ab}e_\mu{}^a\partial_\alpha\delta e_\beta{}^b = \\ &= n^\beta(h^\alpha{}_b+\varepsilon n^\alpha n_b)\partial_\alpha\delta e_\beta{}^b = \\ &= n^\beta e^\alpha{}_b\partial_\alpha\delta e_\beta{}^b \end{aligned} \tag{3.37}$$

where in the first equality we used the definition of the induced metric (2.9), in the second equality $h^{\alpha\beta} = e^{\beta c}h^\alpha{}_c$, in the third equality eq. (3.29) to eliminate the first term and to add a zero term $n^\beta h^\alpha{}_b\partial_\alpha\delta e_\beta{}^b = 0$ and finally, in the last equality we used eq. (3.28). Using the two terms of eqs. (3.36) and (3.37) to express the first term of the surface integral of eq. (3.31), we find

$$\int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}n^\mu g^{\alpha\beta}\partial_\nu\delta g_{\mu\beta} = \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}2n^\mu e^\alpha{}_a\partial_\alpha\delta e_\mu{}^a\,, \tag{3.38}$$

and a similar expression for the second term of the surface integral of eq. (3.31). We therefore obtain the following result

$$16\pi\,\delta S_{EH} = \int_{\mathcal{M}} d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}2n^\mu(e^\alpha{}_a\partial_\alpha\delta e_\mu{}^a - e^\alpha{}_a\partial_\mu\delta e_\alpha{}^a)\,. \tag{3.39}$$

We now verify that the contribution of $S_{torsion}$ of the teleparallel action $S_T$ induces exactly the surface term given in eq. (3.39). Using Gauss' theorem we find

$$16\pi S_{torsion} \equiv \int_{\mathcal{M}} d^4x\,e\,2\nabla_\mu T^{\alpha\mu}{}_\alpha = \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}\,2n_\mu T^{\alpha\mu}{}_\alpha\,. \tag{3.40}$$

Using the fact that $\delta g_{\mu\nu} = \delta e_\mu{}^\nu = \delta n_\mu = 0$ on the boundary, the variation of $S_{torsion}$ is

$$16\pi\delta S_{torsion} = \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}\,2\delta(n_\mu T^{\alpha\mu}{}_\alpha) = \int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}\,2n^\mu e^\alpha{}_a(\partial_\mu\delta e_\alpha{}^a - \partial_\alpha\delta e_\mu{}^a)\,, \tag{3.41}$$

where we used the expression of the torsion tensor given in eq. (3.13).

Combining eq. (3.39) and eq. (3.41) we get

$$16\pi\delta S_T = 16\pi(\delta S_{EH} + \delta S_{torsion}) = \int_{\mathcal{M}} d^4x\,\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}\,, \tag{3.42}$$

indeed the surface term of the Einstein-Hilbert action is exactly cancelled out by the $S_{torsion}$ term of the teleparallel action. Eq. (3.42) tell us that the teleparallel action constitute a well-posed solution for the variational principle of spacetime with boundaries.

Integrating eq. (3.42) we find the teleparallel action of general relativity up to a constant of integration $S_0$

$$S_T = S_{EH} + S_{torsion} + S_0\,, \tag{3.43}$$

note that $S_0$ is independent of the metric $g_{\mu\nu}$ and it should be chosen in order to have the teleparallel action $S_T$ physical at $r\to\infty$ as we did for the $S_{GHY}$ counter term in section 2.2.4.

### 3.3.4 Covariant approach in Riemann-Cartan spacetime

In the previous section we showed that the teleparallel action in Weitzenböck spactime is equivalent to the Einstein-Hilbert action plus the GHY boundary term. Here we want introduce the teleparallel action using a completely covariant approach.

We are going to follow the approach presented by A. Golovnev *et al.* [21] where they used the Lagrange multipliers to impose the condition $\widetilde{R}^{\alpha}{}_{\mu\beta\nu}(\widetilde{\omega}) = 0$ to the teleparallel action in Riemann-Cartan spacetime. In their paper, they directly stated the final result of the variation with respect to the measure $e$ without providing the calculations, here as an exercise, we will go through every single step[8]. Note that these authors use a different notation for the Riemann tensor $R^{\alpha}{}_{\mu\beta\nu}$, torsion scalar $\mathbb{T}$ and contorsion tensor $K^{\alpha}{}_{\mu\nu}$ than the one we are using.

The teleparallel action in Riemann-Cartan spacetime with Lagrange multiplier $\lambda^{\mu\nu}{}_{ab}$ used to impose the condition $\widetilde{R}^{\alpha}{}_{\mu\beta\nu}(\widetilde{\omega}) = 0$ is

$$S_T^{RC} = \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\left(\mathbb{T}(e,\widetilde{\omega}) + \lambda^{\mu\nu}{}_{ab}\widetilde{R}_{\mu\nu}{}^{ab}(\widetilde{\omega})\right)\,, \tag{3.44}$$

where the Lagrange multiplier satisfies the antisymmetry properties $\lambda^{\mu\nu}{}_{ab} = -\lambda^{\nu\mu}{}_{ab} = -\lambda^{\mu\nu}{}_{ba}$.

The variation of the teleparallel action with respect to $\lambda$ yields the desired condition

$$\widetilde{R}_{\mu\nu}{}^{ab}(\widetilde{\omega}) = 0\,, \tag{3.45}$$

which implies $\widetilde{\omega} = 0$, as shown in eq. (3.11).

To perform the variation with respect to the mesure $e = \sqrt{-g}$, we first need to compute the variation of the tetrad field $e^{\mu}{}_a$, the metric $g_{\mu\nu}$ and the torsion tensor $T^{\lambda}{}_{\mu\nu}$. We use the tetrad identity given in eq. (2.88) to compute the variation of the tetrad field $e^{\mu}{}_a$ as follows

$$0 = \delta(\delta^{\mu}_{\nu}) = \delta(e^{\mu}{}_a e_{\nu}{}^a) = \delta e^{\mu}{}_a e_{\nu}{}^a + e^{\mu}{}_b\delta e_{\nu}{}^b \;\Leftrightarrow\; \delta e^{\mu}{}_a = -e^{\mu}{}_b e^{\nu}{}_a\delta e_{\nu}{}^b\,. \tag{3.46}$$

To compute the variation of $e = \sqrt{-g}$ we need to compute the variation of the metric $g_{\mu\nu}$, using eq. (2.90), we obtain

$$\delta g_{\mu\nu} = \eta_{ab}(\delta e_{\mu}{}^a e_{\nu}{}^b + e_{\mu}{}^a\delta e_{\nu}{}^b) = \eta_{ab}(\delta e_{\mu}{}^b e_{\nu}{}^a + e_{\mu}{}^a\delta e_{\nu}{}^b)\,, \tag{3.47}$$

where in the last equality we used the symmetry of $\eta_{ab}$ to rewrite the first term. The variation of the inverse of the metric is found as follows

$$\begin{aligned} 0 &= \delta(\delta^{\mu}_{\lambda}) = \delta(g^{\mu\nu}g_{\nu\lambda}) = \delta g^{\mu\nu}g_{\nu\lambda} + g^{\mu\nu}\delta g_{\nu\lambda} \\ \Leftrightarrow\;\; \delta g^{\mu\nu} &= -g^{\nu\lambda}g^{\mu\sigma}\delta g_{\sigma\lambda} = -\eta_{ab}g^{\nu\lambda}g^{\mu\sigma}(\delta e_{\sigma}{}^b e_{\lambda}{}^a + e_{\sigma}{}^b\delta e_{\lambda}{}^a) = \\ &= -(\delta e_{\sigma}{}^b e^{\nu}{}_b g^{\mu\sigma} + e^{\mu}{}_a g^{\nu\lambda}\delta e_{\lambda}{}^a) = -(e^{\nu}{}_b g^{\mu\lambda} + e^{\mu}{}_b g^{\nu\lambda})\delta e_{\lambda}{}^b\,. \end{aligned} \tag{3.48}$$

[8] These calculations are similar to the steps presented in section IIIB [21], where they derived the equations of motion from the teleparallel action in Weitzenböck spactime. The main difference in the Riemann-Cartan spacatime is the non vanishing spin connection $\widetilde{\omega}$.

We now use the well known identity $\frac{\partial(-g)}{\partial g^{\mu\nu}} = -gg^{\mu\nu}\delta g_{\mu\nu}$ together with eq. (3.47) to calculate

$$\delta e = \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = \frac{1}{2\sqrt{-g}}\frac{\partial(-g)}{\partial g^{\mu\nu}} = \frac{1}{2}eg^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}e(\delta e_\mu{}^b e^\mu{}_b + e^\nu{}_b \delta e_\nu{}^b) = ee^\mu{}_b\delta e_\mu{}^b\,. \tag{3.49}$$

As already mentioned, to be able to compute the variation of the torsion scalar $\mathbb{T}$ we need to compute the variation of the torsion tensor $T^\alpha{}_{\mu\nu}$, therefore varying this tensor we obtain

$$\begin{aligned}\delta T^\alpha{}_{\mu\nu} &= \delta(\widetilde{\Gamma}^\alpha{}_{\mu\nu} - \widetilde{\Gamma}^\alpha{}_{\nu\mu}) = \delta(e^\alpha{}_a\partial_\mu e_\nu{}^a + \widetilde{\omega}_\mu{}^a{}_b e^\alpha{}_a e_\nu{}^b - e^\alpha{}_a\partial_\nu e_\mu{}^a - \widetilde{\omega}_\nu{}^a{}_b e^\alpha{}_a e_\mu{}^b) = \\ &= \delta e^\alpha{}_a(\partial_\mu e_\nu{}^a + \widetilde{\omega}_\mu{}^a{}_b e_\nu{}^b - \partial_\nu e_\mu{}^a - \widetilde{\omega}_\nu{}^a{}_b e_\mu{}^b) + e^\alpha{}_a(\partial_\mu \delta e_\nu{}^a + \widetilde{\omega}_\mu{}^a{}_b \delta e_\nu{}^b - \partial_\nu \delta e_\mu{}^a - \widetilde{\omega}_\nu{}^a{}_b\delta e_\mu{}^b) = \\ &= -e^\alpha{}_b T^\kappa{}_{\mu\nu}\delta e_\kappa{}^b + e^\alpha{}_a(\widetilde{\mathcal{D}}_\mu\delta e_\nu{}^a - \widetilde{\mathcal{D}}_\nu \delta e_\mu{}^a)\,,\end{aligned} \tag{3.50}$$

where in the last equality we used eq. (3.46) to rewrite $\delta e^\alpha{}_a$ and the *Lorenz covariant derivative*, i.e. $\widetilde{\mathcal{D}}_\mu e_\nu{}^a := \partial_\mu e_\nu{}^a + \widetilde{\omega}_\mu{}^a{}_b e_\nu{}^b$ a notation similar to the one given in eq. (2.93), where here we have the spin connection $\widetilde{\omega}_\mu{}^a{}_b$ instead of the affine connection $\widetilde{\Gamma}^\mu{}_{\alpha\beta}$. Furthermore, note that throughout the calculation $\widetilde{\omega} \neq 0$ because we are working in a Riemann-Cartan spacetime.

We can now perform the variation of the following quantities that constitute the torsion scalar $\mathbb{T}$ as given in eq. (3.16):

$$\begin{aligned}\delta(T^\lambda{}_{\mu\nu}T_\lambda{}^{\mu\nu}) &= \delta T^\lambda{}_{\mu\nu}T_\lambda{}^{\mu\nu} + T^\lambda{}_{\mu\nu}\delta T_\lambda{}^{\mu\nu} = \delta T^\lambda{}_{\mu\nu}T_\lambda{}^{\mu\nu} + T^\lambda{}_{\mu\nu}\delta(g_{\lambda\kappa}g^{\mu\alpha}g^{\nu\beta}T^\kappa{}_{\alpha\beta}) = \\ &= \delta T^\lambda{}_{\mu\nu}T_\lambda{}^{\mu\nu} + \delta g_{\lambda\kappa}T^{\kappa\mu\nu}T^\lambda{}_{\mu\nu} + 2\delta g^{\mu\alpha}T_{\lambda\alpha}{}^\nu T^\lambda{}_{\mu\nu} + g_{\lambda\kappa}g^{\mu\alpha}g^{\nu\beta}\delta T^\kappa{}_{\alpha\beta}T^\lambda{}_{\mu\nu} = \\ &= -T^\kappa{}_{\mu\nu}T_\lambda{}^{\mu\nu}e^\lambda{}_b\delta e_\kappa{}^b + e^\lambda{}_a(\widetilde{\mathcal{D}}_\mu\delta e_\nu{}^a - \widetilde{\mathcal{D}}_\nu\delta e_\mu{}^a)T_\lambda{}^{\mu\nu} + \eta_{ab}(\delta e_\lambda{}^b e_\kappa{}^a + \delta e_\kappa{}^b e_\lambda{}^a)T^{\kappa\mu\nu}T^\lambda{}_{\mu\nu} \\ &\quad - 2(g^{\mu\sigma}e^\alpha{}_b + g^{\alpha\sigma}e^\mu{}_b)\delta e_\sigma{}^b T_{\lambda\alpha}{}^\nu T^\lambda{}_{\mu\nu} - T_{\kappa\mu\nu}T^{\sigma\mu\nu}e^\kappa{}_b\delta e_\sigma{}^b + e^\kappa{}_a(\widetilde{\mathcal{D}}_\alpha\delta e_\beta{}^a - \widetilde{\mathcal{D}}_\beta\delta e_\alpha{}^a)T_\kappa{}^{\alpha\beta} = \\ &= -4T^{\lambda\sigma\nu}T_{\lambda\mu\nu}e^\mu{}_b\delta e_\sigma{}^b + 4T_\lambda{}^{\mu\nu}e^\lambda{}_a\widetilde{\mathcal{D}}_\mu\delta e_\nu{}^a\,,\end{aligned} \tag{3.51}$$

$$\begin{aligned}\delta(T^{\mu\nu}{}_\lambda T^\lambda{}_{\mu\nu}) &= \delta T^{\mu\nu}{}_\lambda T^\lambda{}_{\mu\nu} + T^{\mu\nu}{}_\lambda\delta T^\lambda{}_{\mu\nu} = \delta(g^{\nu\rho}T^\mu{}_{\rho\lambda})T^\lambda{}_{\mu\nu} + T^{\mu\nu}{}_\lambda\delta T^\lambda{}_{\mu\nu} = \\ &= \delta g^{\nu\rho}T^\mu{}_{\rho\lambda}T^\lambda{}_{\mu\nu} + g^{\nu\rho}\delta T^\mu{}_{\rho\lambda}T^\lambda{}_{\mu\nu} + T^{\mu\nu}{}_\lambda\delta T^\lambda{}_{\mu\nu} = \\ &= -(g^{\nu\sigma}e^\rho{}_b + e^\nu{}_b g^{\rho\sigma})\delta e_\sigma{}^b T^\mu{}_{\rho\lambda}T^\lambda{}_{\mu\nu} - T^{\kappa\nu}{}_\lambda T^\lambda{}_{\mu\nu}e^\mu{}_b\delta e_\kappa{}^b \\ &\quad + e^\mu{}_a(\widetilde{\mathcal{D}}_\rho\delta e_\lambda{}^a - \widetilde{\mathcal{D}}_\lambda\delta e_\rho{}^a)T^\lambda{}_\mu{}^\rho - T^\kappa{}_{\mu\nu}T^{\mu\nu}{}_\lambda e^\lambda{}_b\delta e_\kappa{}^b + e^\lambda{}_a(\widetilde{\mathcal{D}}_\mu\delta e_\nu{}^a - \widetilde{\mathcal{D}}_\nu\delta e_\mu{}^a)T^{\mu\nu}{}_\lambda = \\ &= 2(T^{\kappa\mu\nu} - T^{\nu\mu\kappa})T_{\mu\lambda\nu}e^\lambda{}_b\delta e_\kappa{}^b + 2e^\mu{}_a(\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a - \widetilde{\mathcal{D}}_\lambda\delta e_\nu{}^a)T^\lambda{}_\mu{}^\nu = \\ &= 2(T^{\kappa\mu\nu} - T^{\nu\mu\kappa})T_{\mu\lambda\nu}e^\lambda{}_b\delta e_\kappa{}^b + 2(T^\lambda{}_\mu{}^\nu - T^\nu{}_\mu{}^\lambda)e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a\,,\end{aligned} \tag{3.52}$$

$$\begin{aligned}\delta(T^\alpha{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma) &= \delta T^\alpha{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma + T^\alpha{}_{\alpha\sigma}\delta T^\nu{}_\nu{}^\sigma = \delta T^\alpha{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma + T^\alpha{}_{\alpha\sigma}\delta(g^{\sigma\kappa}T^\nu{}_{\nu\kappa}) = \\ &= \delta T^\alpha{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma + \delta g^{\sigma\kappa}T^\nu{}_{\nu\kappa}T^\alpha{}_{\alpha\sigma} + g^{\sigma\kappa}\delta T^\nu{}_{\nu\kappa}T^\alpha{}_{\alpha\sigma} = \\ &= -T^\kappa{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma e^\alpha{}_b\delta e_\kappa{}^b + e^\alpha{}_a(\widetilde{\mathcal{D}}_\alpha\delta e_\sigma{}^a - \widetilde{\mathcal{D}}_\sigma\delta e_\alpha{}^a)T^\nu{}_\nu{}^\sigma - (g^{\sigma\lambda}e^\kappa{}_b + g^{\kappa\lambda}e^\sigma{}_b)\delta e_\lambda{}^b T^\nu{}_{\nu\kappa}T^\alpha{}_{\alpha\sigma} \\ &\quad - T^\rho{}_{\nu\kappa}T^\alpha{}_\alpha{}^\kappa e^\nu{}_b\delta e_\rho{}^b + e^\nu{}_a(\widetilde{\mathcal{D}}_\nu\delta e_\kappa{}^a - \widetilde{\mathcal{D}}_\kappa\delta e_\nu{}^a)T^\alpha{}_\alpha{}^\kappa = \\ &= -2(T^\kappa{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma + T^\lambda{}_\lambda{}^\kappa T^\nu{}_{\nu\alpha})e^\alpha{}_b\delta e_\kappa{}^b + 2(e^\alpha{}_a T^\nu{}_\nu{}^\sigma - e^\sigma{}_a T^\nu{}_\nu{}^\alpha)\widetilde{\mathcal{D}}_\alpha\delta e_\sigma{}^a\end{aligned} \tag{3.53}$$

where in the last equality of all these variations we used the antisymmetry of the torsion tensor $T^\alpha{}_{\mu\nu} = -T^\alpha{}_{\nu\mu}$ to add similar terms together.

We have therefore calculated all the terms we need to compute the variation of the teleparallel action, plugging them all together, we get

$$\delta S_T^{RC} = \frac{1}{16\pi}\int d^4x \left\{ e\left(-\frac{1}{4}\delta(T^\lambda{}_{\mu\nu}T_\lambda{}^{\mu\nu}) + \frac{1}{2}\delta(T^{\mu\nu}{}_\lambda T^\lambda{}_{\mu\nu}) + \delta(T^\alpha{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma)\right) + \left(\mathbb{T}(e,\widetilde{\omega}) + \lambda^{\mu\nu}{}_{ab}R_{\mu\nu}{}^{ab}(\widetilde{\omega})\right)\delta e \right\}. \tag{3.54}$$

Before proceeding with the simplification of the terms in the brackets, we define the *superpotential* $S^{\alpha\mu\nu}$ which will simplify a lot the expressions we are going to calculate

$$S^{\alpha\mu\nu} := K^{\mu\nu\alpha} + g^{\alpha\mu}T^{\beta\nu}{}_\beta - g^{\alpha\nu}T^{\beta\mu}{}_\beta\,, \tag{3.55}$$

which satisfy the antisymmetry condition $S^{\alpha\mu\nu} = -S^{\alpha\nu\mu}$, because the contorsion tensor is antisymmetric with respect to the first two indices.

We want to group similar terms in the variation of the teleparallel action. We begin with terms proportional to $\widetilde{\mathcal{D}}_\mu \delta e_\nu{}^a$. The second term of eq. (3.51) combines with the second term of eq. (3.52) to form

$$\begin{aligned}-\frac{1}{4}4T_\lambda{}^{\nu\mu}e^\lambda{}_a\widetilde{\mathcal{D}}_\nu\delta e_\mu{}^a + \frac{1}{2}2(T^\lambda{}_\mu{}^\nu - T^\nu{}_\mu{}^\lambda)e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a &= (T_\mu{}^{\lambda\nu} - T^{\lambda\nu}{}_\mu + T^{\nu\lambda}{}_\mu)e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a = \\ &= 2K^{\lambda\nu}{}_\mu e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a\,,\end{aligned} \tag{3.56}$$

where we used the antisymmetry of the torsion tensor and the definition of the contorsion tensor given in eq. (3.8). Combining this term with the second term of eq. (3.53), we get

$$\begin{aligned}2K^{\lambda\nu}{}_\mu e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a + 2(e^\nu{}_aT^\alpha{}_\alpha{}^\lambda - e^\lambda{}_aT^\alpha{}_\alpha{}^\nu)\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a &= 2(K^{\lambda\nu}{}_\mu e^\mu{}_a + e^\nu{}_aT^\alpha{}_\alpha{}^\lambda - e^\lambda{}_aT^\alpha{}_\alpha{}^\nu)\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a = \\ &= 2S_\mu{}^{\lambda\nu}e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a\,,\end{aligned} \tag{3.57}$$

where we used the superpotential given in eq. (3.55) to substitute

$$S_\mu{}^{\lambda\nu}e^\mu{}_a = K^{\lambda\nu}{}_\mu e^\mu{}_a + g_\mu{}^\lambda e^\mu{}_aT^{\alpha\nu}{}_\alpha - g_\mu{}^\nu e^\mu{}_aT^{\alpha\lambda}{}_\alpha = K^{\lambda\nu}{}_\mu e^\mu{}_a - e^\lambda{}_aT^\alpha{}_\alpha{}^\nu + e^\nu{}_aT^\alpha{}_\alpha{}^\lambda\,. \tag{3.58}$$

We proceed by combining the terms proportional to $e^\lambda{}_b\delta e_\kappa{}^b$, the first term of eq. (3.51) combines with the second term of eq. (3.52) to form

$$\begin{aligned}\frac{1}{4}4T^{\lambda\sigma\nu}T_{\lambda\mu\nu}e^\mu{}_b\delta e_\sigma{}^b + \frac{1}{2}2(T^{\kappa\mu\nu} - T^{\nu\mu\kappa})T_{\mu\lambda\nu}e^\lambda{}_b\delta e_\kappa{}^b &= (T^{\lambda\kappa\nu} + T^{\kappa\lambda\nu} - T^{\nu\lambda\kappa})T_{\lambda\mu\nu}e^\mu{}_b\delta e_\kappa{}^b = \\ &= 2K^{\kappa\nu\lambda}T_{\lambda\mu\nu}e^\mu{}_b\delta e_\kappa{}^b\,,\end{aligned} \tag{3.59}$$

where in the last equality we used the antisymmetry of the torsion tensor to write the last two terms in the brackets as $T^{\kappa\lambda\nu} = -T^{\kappa\nu\lambda}$, $-T^{\nu\lambda\kappa} = T^{\nu\kappa\lambda}$ and the definition of the contorsion tensor given in eq. (3.8). Combing this term with the second term of eq. (3.53), we get

$$\begin{aligned}2K^{\kappa\nu\lambda}T_{\lambda\mu\nu}e^\mu{}_b\delta e_\kappa{}^b - 2(T^\kappa{}_{\alpha\sigma}T^\nu{}_\nu{}^\sigma + T^\lambda{}_\lambda{}^\kappa T^\nu{}_{\nu\alpha})e^\alpha{}_b\delta e_\kappa{}^b &= 2(K^{\kappa\nu\lambda}T_{\lambda\mu\nu} - T^\kappa{}_{\mu\sigma}T^\rho{}_\rho{}^\sigma - T^\rho{}_\rho{}^\kappa T^\delta{}_{\delta\mu})e^\mu{}_b\delta e_\kappa{}^b = \\ &= 2S^{\lambda\kappa\nu}T_{\lambda\mu\nu}e^\mu{}_b\delta e_\kappa{}^b\,,\end{aligned} \tag{3.60}$$

where we used the superpotential given in eq. (3.55) to substitute

$$S^{\lambda\kappa\nu}T_{\lambda\mu\nu} = (K^{\kappa\nu\lambda} + g^{\lambda\kappa}T^{\rho\nu}{}_\rho - g^{\lambda\nu}T^{\rho\kappa}{}_\rho)T_{\lambda\mu\nu} = K^{\kappa\nu\lambda}T_{\lambda\nu\mu} - T^\rho{}_\rho{}^\nu T^\kappa{}_{\mu\nu} - T^\rho{}_\rho{}^\kappa T^\nu{}_{\nu\mu}\,. \tag{3.61}$$

Summarizing all the terms, the variation with respect to the measure $e$ of the teleparallel action yields

$$\delta S_T^{RC} = \frac{1}{16\pi}\int d^4x\, e\left(2S^{\lambda\kappa\nu}T_{\lambda\mu\nu}e^{\mu}{}_b\delta e_\kappa{}^b + 2S_\mu{}^{\lambda\nu}e^\mu{}_a\widetilde{\mathcal{D}}_\nu\delta e_\lambda{}^a + \big(\mathbb{T}(e,\widetilde{\omega}) + \lambda^{\mu\nu}{}_{ab}\widetilde{R}_{\mu\nu}{}^{ab}(\widetilde{\omega})\big)e^\mu{}_b\delta e_\mu{}^b\right). \tag{3.62}$$

Integrating by part the second term in the brackets of eq. (3.62), we get

$$\begin{aligned} 2\delta e_\lambda{}^a\widetilde{\mathcal{D}}_\nu(eS_\mu{}^{\lambda\nu}e^\mu{}_a) &= 2\delta e_\lambda{}^a(\partial_\nu(eS_\mu{}^{\lambda\nu}e^\mu{}_a) - \widetilde{\omega}_\nu{}^b{}_a eS_\mu{}^{\lambda\nu}e^\mu{}_b) = \\ &= 2\delta e_\lambda{}^a\Big(\big(\partial_\nu(eS_\mu{}^{\lambda\nu}e^\mu{}_a) - \omega_\nu{}^b{}_a eS_\mu{}^{\lambda\nu}e^\mu{}_b\big) + \big(\omega_\nu{}^b{}_a eS_\mu{}^{\lambda\nu}e^\mu{}_b - \widetilde{\omega}_\nu{}^b{}_a eS_\mu{}^{\lambda\nu}e^\mu{}_b\big)\Big) = \\ &= 2e\big(\nabla_\nu S_\mu{}^{\lambda\nu} - K^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\nu}\big)e^\mu{}_a\delta e_\lambda{}^a\,, \end{aligned} \tag{3.63}$$

where we used the antisymmetry of the superpotential, i.e. $S^{\alpha\mu\nu} = -S^{\alpha\nu\mu}$, and the tetrad postulate of eq. (2.94) to write

$$\begin{aligned} (\nabla_\nu S_\mu{}^{\alpha\beta})e^\mu{}_b = (\nabla_\nu S_\mu{}^{\alpha\beta}e^\mu{}_b) = \nabla_\nu S_b{}^{\alpha\beta} &= \partial_\nu S_a{}^{\alpha\beta} + \Gamma^\alpha{}_{\sigma\nu}S_a{}^{\sigma\beta} + \Gamma^\beta{}_{\sigma\nu}S_a{}^{\alpha\sigma} - \omega_\nu{}^b{}_a S_b{}^{\alpha\beta} = \\ = \partial_\nu S_a{}^{\alpha\beta} - \omega_\nu{}^b{}_a S_b{}^{\alpha\beta} &= \partial_\nu(S_\mu{}^{\alpha\beta}e^\mu{}_a) - \omega_\nu{}^b{}_a S_\mu{}^{\alpha\beta}e^\mu{}_b\,, \end{aligned} \tag{3.64}$$

and

$$K^\mu{}_{\alpha\beta} = \widetilde{\Gamma}^\mu{}_{\alpha\beta} - \Gamma^\mu_{\alpha\beta} = (\widetilde{\omega}_\beta{}^b{}_a - \omega_\beta{}^b{}_a)e^\mu{}_b e_\alpha{}^a\,, \tag{3.65}$$

note that we are not using the same convention for the connection as in section 2.4.2 where we had $\Gamma^\mu{}_{\alpha\beta} = e^\mu{}_a(\partial_\alpha e_\beta{}^a + \omega_\alpha{}^a{}_b e_\beta{}^b)$, because here we defined the contorsion tensor such that it is antisymmetric with respect to the first two indices, i.e. $K^{\mu\alpha\beta} = -K^{\alpha\mu\beta}$, while in eq. (2.93) we defined the spin connection with antisymmetry in the last two indices, see eq. (2.112), i.e. $\omega_\alpha{}^{ab} = -\omega_\alpha{}^{ab}$. Taking into account this consideration, the right expression that relates the contorsion tensor and the spin connection, according to our notation, is the one given in eq. (3.65) where the third index of $K^\mu{}_{\alpha\beta}e_\mu{}^a e^\alpha{}_b = K^a{}_{b\beta}$ correspond to the first index of $\omega_\beta{}^a{}_b$.

Therefore the variational principle of the teleparallel action in Riemann-Cartan spacetime yields to

$$\begin{aligned} \delta S_T^{RC} = \frac{1}{8\pi}\int d^4x\, e\Big(-\nabla_\nu S_\mu{}^{\lambda\nu} + S^{\sigma\lambda\nu}(T_{\sigma\mu\nu} + K_{\sigma\mu\nu}) + \frac{1}{2}\big(\mathbb{T}(e,\widetilde{\omega}) + \lambda^{\mu\nu}{}_{ab}R_{\mu\nu}{}^{ab}(\widetilde{\omega})\big)\delta^\lambda_\mu\Big)e^\mu{}_b\delta e_\lambda{}^b \stackrel{!}{=} 0 \\ \Leftrightarrow 0 = \nabla_\nu S_\mu{}^{\lambda\nu} - S^{\sigma\lambda\nu}(T_{\sigma\mu\nu} + K_{\sigma\mu\nu}) - \frac{1}{2}\big(\mathbb{T}(e,\widetilde{\omega}) + \lambda^{\mu\nu}{}_{ab}R_{\mu\nu}{}^{ab}(\widetilde{\omega})\big)\delta^\lambda_\mu\,. \end{aligned} \tag{3.66}$$

Enforcing the condition obtained from the variation of the teleparallel action with respect to $\lambda$, i.e. $R_{\mu\nu}{}^{ab}(\widetilde{\omega}) = 0 \Rightarrow \widetilde{\omega} = 0$ as show in eq. (3.11) and using $T_{\sigma\mu\nu} + K_{\sigma\mu\nu} = K_{\sigma\nu\mu}$, we obtain the following field equation

$$\nabla_\nu S_\mu{}^{\lambda\nu} - S^{\sigma\lambda\nu}K_{\sigma\nu\mu} - \frac{1}{2}\mathbb{T}(e)\delta^\lambda_\mu = 0\,, \tag{3.67}$$

which is equivalent to the Einstein field equations as we are going to show in the following section. Note that this result is equivalent to the one given by Gikivbev *et al.* [21], remember when comparing it with our own that they are using a different notation of the contortion tensor $K^\mu{}_{\alpha\beta}$ and sign convention of the torsion scalar $\mathbb{T}$ than the one we are using.

Finally, the variation of the teleparallel action with respect to the spin connection $\widetilde{\omega}_\alpha{}^b{}_a$ gives, after partial integration[9],

$$\widetilde{\mathcal{D}}_\mu(e\lambda^{\mu\nu}{}_{ab}) = 0\,, \tag{3.68}$$

this is an equation for the Lagrange multiplier $\lambda_{\mu\nu}{}^{ab}$. As the Lagrange multiplier does not enter in the equation of motion given in eq. (3.67), we do not need to consider it any further.

### 3.3.5 Einstein field equations

Here we want to show that the field equations calculated in eq. (3.67) are equivalent to the conventional general relativity by substituting eq. (3.18), rewritten here for convenience,

$$R^\alpha{}_{\mu\beta\nu} = \nabla_\nu K^\alpha{}_{\mu\beta} - \nabla_\beta K^\alpha{}_{\mu\nu} + K^\alpha{}_{\lambda\nu}K^\lambda{}_{\mu\beta} - K^\alpha{}_{\lambda\beta}K^\lambda{}_{\mu\nu}\,, \tag{3.69}$$

into the Einstein fields equations

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R = 0\,. \tag{3.70}$$

Inserting the Riemann tensor and the expression calculated in eq. (3.24) for the Ricci scalar, we find

$$0 = G^\mu{}_\nu = \nabla_\nu K^{\alpha\mu}{}_\alpha - \nabla_\alpha K^{\alpha\mu}{}_\nu + K^\alpha{}_{\lambda\nu}K^{\lambda\mu}{}_\alpha - K^\alpha{}_{\lambda\alpha}K^{\lambda\mu}{}_\nu - \frac{1}{2}\delta^\mu_\nu\left(\mathbb{T} - 2\nabla_\beta T^{\alpha\beta}{}_\alpha\right)\,. \tag{3.71}$$

The first two terms of eq. (3.71) together with the last one, can be rewritten as

$$\begin{aligned}\nabla_\nu K^{\alpha\mu}{}_\alpha - \nabla_\alpha K^{\alpha\mu}{}_\nu - \delta^\mu_\nu\nabla_\beta T^{\alpha\beta}{}_\alpha &= \frac{1}{2}\nabla_\nu(T^{\mu\alpha}{}_\alpha + T_\alpha{}^{\alpha\mu} - T^{\alpha\mu}{}_\alpha) - \nabla_\alpha K^{\alpha\mu}{}_\nu + \delta^\mu_\nu\nabla_\beta T^{\alpha\beta}{}_\alpha = \\ &= -\nabla_\alpha K^{\alpha\mu}{}_\nu - \nabla_\nu T^{\beta\mu}{}_\beta + \delta^\mu_\nu\nabla_\alpha T^{\beta\alpha}{}_\beta = \\ &= \nabla_\alpha S_\nu{}^{\mu\alpha}\,,\end{aligned} \tag{3.72}$$

where in the second equality we used the antisymmetry of the torsion tensor $T^\alpha{}_{\mu\nu} = -T^\alpha{}_{\nu\mu}$ to rewrite $T^{\mu\alpha}{}_\alpha = 0$ and $T_\alpha{}^{\alpha\mu} = -T_\alpha{}^{\mu\alpha} = -T^{\alpha\mu}{}_\alpha$ and in the last equality, we used the definition of the superpotatial given in eq. (3.55) to get

$$\nabla_\alpha S_\nu{}^{\mu\alpha} = \nabla_\alpha(K^{\mu\alpha}{}_\nu + \delta^\mu_\nu T^{\beta\alpha}{}_\beta - \delta^\alpha_\nu T^{\beta\mu}{}_\beta) = -\nabla_\alpha K^{\alpha\mu}{}_\nu + \delta^\mu_\nu\nabla_\alpha T^{\beta\alpha}{}_\beta - \nabla_\nu T^{\beta\mu}{}_\beta\,, \tag{3.73}$$

where we used the antisymmetry property of the torsion tensor $K^{\mu\alpha}{}_\nu = -K^{\alpha\mu}{}_\nu$.

The third and fourth terms of eq. (3.71) can be rewritten as

$$\begin{aligned}K^\alpha{}_{\lambda\nu}K^{\lambda\mu}{}_\alpha - K^\alpha{}_{\lambda\alpha}K^{\lambda\mu}{}_\nu &= K_{\alpha\lambda\nu}K^{\lambda\mu\alpha} - K^{\alpha\lambda}{}_\alpha K_\lambda{}^\mu{}_\nu = \\ &= K_{\alpha\lambda\nu}(-K^{\mu\lambda\alpha} - g^{\alpha\mu}T^{\beta\lambda}{}_\beta + g^{\alpha\lambda}T^{\beta\mu}{}_\beta) + K^\mu{}_{\lambda\nu}T^{\beta\lambda}{}_\beta \\ &\quad - K_\alpha{}^\alpha{}_\nu T^{\beta\mu}{}_\beta - \frac{1}{2}K_\lambda{}^\mu{}_\nu(T^{\lambda\alpha}{}_\alpha + T_\alpha{}^{\alpha\lambda} - T^{\alpha\lambda}{}_\alpha) = \\ &= -K_{\alpha\lambda\nu}S^{\alpha\mu\lambda}\end{aligned} \tag{3.74}$$

[9] For more details see section IVC of [21].

where in the second equality we added two terms which sum up to zero, used the antisymmetry of the contorsion tensor to rewrite $K^{\lambda\mu\alpha} = -K^{\mu\lambda\alpha}$, the antisymmetry of the torsion tensor to get $T^{\lambda\alpha}{}_{\alpha} = 0$ and $T_{\alpha}{}^{\alpha\lambda} = -T_{\alpha}{}^{\lambda\alpha} = -T^{\alpha\lambda}{}_{\alpha}$, and finally, in the third equality, we used $K_{\lambda}{}^{\mu}{}_{\nu} = -K^{\mu}{}_{\lambda\nu}$ and $K_{\alpha}{}^{\alpha}{}_{\nu} = \frac{1}{2}(T^{\alpha}{}_{\alpha\nu} + T_{\nu\alpha}{}^{\alpha} - T_{\alpha}{}^{\alpha}{}_{\nu}) = 0$.

Summarizing all the terms, we obtain the following Einstein field equation

$$\nabla_{\alpha} S_{\nu}{}^{\mu\alpha} - S^{\alpha\mu\lambda} K_{\alpha\lambda\nu} - \frac{1}{2}\mathbb{T}(e)\delta^{\mu}_{\nu} = 0\,, \tag{3.75}$$

which indeed is the same as the one gotten varying the the teleparallel action with the Lagrange multiplier approach obtained in eq. (3.67).

## 3.4 Symmetric teleparallel general relativity

### 3.4.1 Symmetric teleparallel action

The symmetric teleparallel spacetime has non-vanishing metricity, i.e. $Q_{\alpha\mu\nu} = \widetilde{\nabla}_{\alpha} g_{\mu\nu} \neq 0$, and vanishing curvature $R^{\alpha}{}_{\mu\beta\nu}$ and torsion $T^{\alpha}{}_{\mu\nu}$. The connection describing this spacetime is obtained using eq. (3.6) as follows

$$\widetilde{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}\,. \tag{3.76}$$

We define the symmetric teleparallel action as

$$S_Q := \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\mathbb{Q}\,, \tag{3.77}$$

where $\mathbb{Q}$ is the non-metricity scalar defined as

$$\mathbb{Q} \equiv -\frac{1}{4}Q_{\alpha\beta\mu}Q^{\alpha\beta\mu} + \frac{1}{2}Q_{\alpha\beta\mu}Q^{\beta\mu\alpha} + \frac{1}{4}Q_{\mu}{}^{\alpha}{}_{\alpha}Q^{\mu\sigma}{}_{\sigma} - \frac{1}{2}Q_{\mu}{}^{\alpha}{}_{\alpha}Q_{\sigma}{}^{\mu\sigma}\,, \tag{3.78}$$

note that this quantity is in invariant under local general linear transformations and translational symmetry.

As explained by A. Conroy and T. Koivisto [22], the vanishing curvature imposes the connection to be pure inertial, meaning that it differs only by a general linear transformation $J^{\alpha}{}_{\beta}$ from the trivial connection or "coincident gauge", i.e. $\widetilde{\Gamma}^{\alpha}{}_{\mu\nu} = (J^{-1})^{\alpha}{}_{\beta}\partial_{\mu}J^{\beta}{}_{\nu}$, where $(J^{-1})^{\alpha}{}_{\beta}$ are the components of the inverse matrix of the linear transformation $J^{\alpha}{}_{\beta}$. Furthermore, the vanishing torsion further simplified the connection because

$$T^{\alpha}{}_{\mu\nu} = \widetilde{\Gamma}^{\alpha}{}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}{}_{\nu\mu} = (J^{-1})^{\alpha}{}_{\beta}(\partial_{\mu}J^{\beta}{}_{\nu} - \partial_{\nu}J^{\beta}{}_{\mu}) = 0 \tag{3.79}$$

has solution $J^{\alpha}{}_{\beta} = \partial_{\beta}\xi^{\alpha}$ where $\xi^{\alpha}$ is a vector in the tangent space. Note that the translational symmetry of the non-metricity scalar allows to chose $\xi^{\alpha}$, i.e. the gauge condition, such that the connection $\widetilde{\Gamma}^{\alpha}{}_{\mu\nu}$ vanishes completely. A trivial connection implies that all points in spacetime are equivalent, therefore this formulation of general relativity is often called *coincident*. For the gauge with vanishing connection the non metricity scalar $\mathbb{Q}$ becomes, as shown by J. M. Nester and H.-J. Yo [23],

$$\mathbb{Q} = g^{\mu\nu}\left(\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\nu\alpha} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu}\right)\,, \tag{3.80}$$

where $\Gamma^{\alpha}_{\mu\nu}$ are the Christoffel symbols. This result tell us that the symmetric teleparallel action is equivalent to the Einstein-Hilbert action corrected with the GHY boundary term.

### 3.4.2 The non-metricity scalar

In this section we are going to relate the Ricci scalar of the Riemann spacetime to the non-metricity scalar $\mathbb{Q}$ following a similar procedure as we did in section 3.3.2 for the torsion scalar.

Using the fact that the Riemann tensor of the symmetric teleparallel spacetime vanishes and the definition of the symmetric teleparallel connection given eq. (3.76), we obtain

$$\begin{aligned}
0 = \widetilde{R}^{\alpha}{}_{\mu\beta\nu}(\widetilde{\Gamma}) &= \widetilde{\Gamma}^{\alpha}_{\mu\nu,\beta} - \widetilde{\Gamma}^{\alpha}_{\mu\beta,\nu} + \widetilde{\Gamma}^{\alpha}_{\lambda\beta}\widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\alpha}_{\lambda\nu}\widetilde{\Gamma}^{\lambda}_{\mu\beta} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + L^{\alpha}{}_{\mu\nu,\beta} - L^{\alpha}{}_{\mu\beta,\nu} + L^{\alpha}{}_{\lambda\beta}L^{\lambda}{}_{\mu\nu} - L^{\alpha}{}_{\lambda\nu}L^{\lambda}{}_{\mu\beta} \\
&\qquad + \Gamma^{\alpha}_{\lambda\beta}L^{\lambda}{}_{\mu\nu} - \Gamma^{\alpha}_{\lambda\nu}L^{\lambda}{}_{\mu\beta} + \Gamma^{\lambda}_{\mu\nu}L^{\alpha}{}_{\lambda\beta} - \Gamma^{\lambda}_{\mu\beta}L^{\alpha}{}_{\lambda\nu} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + (\partial_\beta L^{\alpha}{}_{\mu\nu} + \Gamma^{\alpha}_{\lambda\beta}L^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}_{\mu\beta}L^{\alpha}{}_{\lambda\nu} - \Gamma^{\lambda}_{\nu\beta}L^{\alpha}{}_{\mu\lambda}) \\
&\qquad - (\partial_\nu L^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}_{\lambda\nu}L^{\lambda}{}_{\mu\beta} - \Gamma^{\lambda}_{\mu\nu}L^{\alpha}{}_{\lambda\beta} - \Gamma^{\lambda}_{\beta\nu}L^{\alpha}{}_{\mu\lambda}) \\
&\qquad + L^{\alpha}{}_{\lambda\beta}L^{\lambda}{}_{\mu\nu} - L^{\alpha}{}_{\lambda\nu}L^{\lambda}{}_{\mu\beta} = \\
&= R^{\alpha}{}_{\mu\beta\nu}(\Gamma) + \nabla_\beta L^{\alpha}{}_{\mu\nu} - \nabla_\nu L^{\alpha}{}_{\mu\beta} + L^{\alpha}{}_{\lambda\beta}L^{\lambda}{}_{\mu\nu} - L^{\alpha}{}_{\lambda\nu}L^{\lambda}{}_{\mu\beta}\,.
\end{aligned} \tag{3.81}$$

Rearranging the terms of eq. (3.81) and taking the trace of the Ricci tensor we obtain the following expression for the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^{\alpha}{}_{\mu\alpha\nu} = \nabla_\mu L^{\alpha\mu}{}_{\alpha} - \nabla_\alpha L^{\alpha\mu}{}_{\mu} + L^{\alpha}{}_{\sigma\nu}L^{\sigma\nu}{}_{\alpha} - L^{\alpha}{}_{\sigma\alpha}L^{\sigma\nu}{}_{\nu}\,. \tag{3.82}$$

The first two terms in eq. (3.82) can be rewritten using the definition of the contortion tensor given in eq. (3.9) as

$$\begin{aligned}
\nabla_\mu L^{\alpha\mu}{}_{\alpha} - \nabla_\alpha L^{\alpha\mu}{}_{\mu} &= \frac{1}{2}\nabla_\mu\Big(Q^{\alpha\mu}{}_{\alpha} - Q^{\mu\alpha}{}_{\alpha} - Q_{\alpha}{}^{\alpha\mu}\Big) - \frac{1}{2}\nabla_\alpha\Big(Q^{\alpha\mu}{}_{\mu} - Q^{\mu\alpha}{}_{\mu} - Q_{\mu}{}^{\alpha\mu}\Big) = \\
&= \nabla_\mu Q^{\alpha\mu}{}_{\alpha} - \nabla_\mu Q^{\mu\alpha}{}_{\alpha} = \nabla_\mu(Q^{\alpha\mu}{}_{\alpha} - Q^{\mu\alpha}{}_{\alpha})\,,
\end{aligned} \tag{3.83}$$

where we used the symmetry of the non-metricity, i.e. $Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} = \nabla_\alpha g_{\nu\mu} = Q_{\alpha\nu\mu}$ and we replaced the dummy indices to combine similar terms.

The third term in eq. (3.82) is rewritten as follows

$$\begin{aligned}
L^{\alpha}{}_{\sigma\nu}L^{\sigma\nu}{}_{\alpha} &= \frac{1}{4}\Big(Q^{\alpha}{}_{\sigma\nu} - Q_{\sigma}{}^{\alpha}{}_{\nu} - Q_{\nu}{}^{\alpha}{}_{\sigma}\Big)\Big(Q^{\sigma\nu}{}_{\alpha} - Q^{\nu\sigma}{}_{\alpha} - Q_{\alpha}{}^{\sigma\nu}\Big) = \\
&= \frac{1}{4}\Big(-Q^{\alpha}{}_{\sigma\nu}Q^{\nu\sigma}{}_{\alpha} - Q_{\sigma}{}^{\alpha}{}_{\nu}Q^{\sigma\nu}{}_{\alpha} + Q_{\sigma}{}^{\alpha}{}_{\nu}Q^{\nu\sigma}{}_{\alpha} + Q_{\sigma}{}^{\alpha}{}_{\nu}Q_{\alpha}{}^{\sigma\nu} - Q_{\nu}{}^{\alpha}{}_{\sigma}Q^{\sigma\nu}{}_{\alpha} + Q_{\nu}{}^{\alpha}{}_{\sigma}Q^{\nu\sigma}{}_{\alpha}\Big) \\
&\quad - \frac{1}{4}Q^{\alpha}{}_{\sigma\nu}Q_{\alpha}{}^{\sigma\nu} + \frac{1}{4}Q^{\alpha}{}_{\sigma\nu}Q^{\sigma\nu}{}_{\alpha} + \frac{1}{4}Q_{\nu}{}^{\alpha}{}_{\sigma}Q_{\alpha}{}^{\sigma\nu} = \\
&= -\frac{1}{4}Q_{\alpha\sigma\nu}Q^{\alpha\sigma\nu} + \frac{1}{2}Q_{\alpha\sigma\nu}Q^{\sigma\nu\alpha}\,.
\end{aligned} \tag{3.84}$$

where in the last equality we used the symmetry of the non-metricity and replaced some dummy indices to rewrite the last three terms in the brackets as follows $Q_{\sigma}{}^{\alpha}{}_{\nu}Q_{\alpha}{}^{\sigma\nu} = Q_{\sigma\nu}{}^{\alpha}Q_{\alpha}{}^{\nu\sigma} = Q_{\nu\sigma}{}^{\alpha}Q_{\alpha}{}^{\sigma\nu} = Q^{\alpha}{}_{\sigma\nu}Q^{\nu\sigma}{}_{\alpha}$; $-Q_{\nu}{}^{\alpha}{}_{\sigma}Q^{\sigma\nu}{}_{\alpha} = -Q_{\nu\sigma}{}^{\alpha}Q^{\sigma}{}_{\alpha}{}^{\nu} = -Q_{\sigma}{}^{\alpha}{}_{\nu}Q^{\nu\sigma}{}_{\alpha}$ and $Q_{\nu}{}^{\alpha}{}_{\sigma}Q^{\nu\sigma}{}_{\alpha} = Q_{\nu\sigma}{}^{\alpha}Q^{\nu}{}_{\alpha}{}^{\sigma} = Q_{\sigma}{}^{\alpha}{}_{\nu}Q^{\sigma\nu}{}_{\alpha}$ and we see that they all compensate each others.

The fourth term in eq. (3.82) is rewritten as follows

$$
\begin{aligned}
-L^{\alpha}{}_{\sigma\alpha}L^{\sigma\nu}{}_{\nu} &= -\frac{1}{4}\Big(Q^{\alpha}{}_{\sigma\alpha} - Q_{\sigma}{}^{\alpha}{}_{\alpha} - Q_{\alpha}{}^{\alpha}{}_{\sigma}\Big)\Big(Q^{\sigma\nu}{}_{\nu} - Q^{\nu\sigma}{}_{\nu} - Q_{\nu}{}^{\sigma\nu}\Big) = \\
&= \frac{1}{4}\Big(Q^{\alpha}{}_{\sigma\alpha}Q^{\nu\sigma}{}_{\nu} + Q^{\alpha}{}_{\sigma\alpha}Q_{\nu}{}^{\sigma\nu} - Q_{\sigma}{}^{\alpha}{}_{\alpha}Q_{\nu}{}^{\sigma\nu} + Q_{\alpha}{}^{\alpha}{}_{\sigma}Q^{\sigma\nu}{}_{\nu} - Q_{\alpha}{}^{\alpha}{}_{\sigma}Q^{\nu\sigma}{}_{\nu} - Q_{\alpha}{}^{\alpha}{}_{\sigma}Q_{\nu}{}^{\sigma\nu}\Big) \\
&\quad + \frac{1}{4}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q^{\sigma\nu}{}_{\nu} - \frac{1}{4}Q^{\alpha}{}_{\sigma\alpha}Q^{\sigma\nu}{}_{\nu} - \frac{1}{4}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q^{\nu\sigma}{}_{\nu} = \\
&= \frac{1}{4}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q^{\sigma\nu}{}_{\nu} - \frac{1}{2}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q_{\nu}{}^{\sigma\nu}\,,
\end{aligned}
\tag{3.85}
$$

where in the last equality we used again the symmetry of the non-metricity and replaced some dummy indices to rewrite the last three terms in the brackets as follows $Q_{\alpha}{}^{\alpha}{}_{\sigma}Q^{\sigma\nu}{}_{\nu} = Q_{\alpha\sigma}{}^{\alpha}Q^{\sigma\nu}{}_{\nu} = Q_{\sigma}{}^{\alpha}{}_{\alpha}Q_{\nu}{}^{\sigma\nu}$; $-Q_{\alpha}{}^{\alpha}{}_{\sigma}Q^{\nu\sigma}{}_{\nu} = -Q^{\alpha}{}_{\sigma\alpha}Q_{\nu}{}^{\sigma\nu}$ and $-Q_{\alpha}{}^{\alpha}{}_{\sigma}Q_{\nu}{}^{\sigma\nu} = -Q^{\alpha}{}_{\sigma\alpha}Q^{\nu\sigma}{}_{\nu}$ and we see that they all compensate each others.

The third and fourth term of eq. (3.82) can be rewritten as

$$
L^{\alpha}{}_{\sigma\nu}L^{\sigma\nu}{}_{\alpha} - L^{\alpha}{}_{\sigma\alpha}L^{\sigma\nu}{}_{\nu} = -\frac{1}{4}Q_{\alpha\sigma\nu}Q^{\alpha\sigma\nu} + \frac{1}{2}Q_{\alpha\sigma\nu}Q^{\sigma\nu\alpha} + \frac{1}{4}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q^{\sigma\nu}{}_{\nu} - \frac{1}{2}Q_{\sigma}{}^{\alpha}{}_{\alpha}Q_{\nu}{}^{\sigma\nu} \equiv \mathbb{Q}\,. \tag{3.86}
$$

Plugging the results of eqs. (3.83) and (3.86) in eq. (3.82), we obtain

$$
R = \mathbb{Q} + \nabla_{\mu}(Q^{\alpha\mu}{}_{\alpha} - Q^{\mu\alpha}{}_{\alpha}) \;\Leftrightarrow\; \mathbb{Q} = R + \nabla_{\mu}(Q^{\mu\alpha}{}_{\alpha} - Q_{\alpha}{}^{\mu\alpha})\,, \tag{3.87}
$$

this is a useful relation which relates the non-metricity scalar of the symmetric teleprallel theory with the Ricci scalar of the conventional general relativity.

### 3.4.3 Covariant approach

As one could imagine, the covariant approach used in section 3.3.4 for teleparallel general relativity can be applied also to symmetric teleparallel general relativity. The main idea is to impose the two conditions of vanishing Riemann curvature $R^{\alpha}{}_{\mu\beta\nu}$ and vanishing torsion $T^{\mu}{}_{\alpha\beta}$ using the Lagrange multiplier method. Here we will not go through the explicit variation of the action but we will summarize the results presented in the paper of J. B. Jiménez, L. Heisemberg and T. Koivisto [24] which one can verify using tensor manipulation software such as for example the package xAct for Mathematica.

The symmetric teleparallel action is defined as

$$
S_Q = \int_{\partial\mathcal{M}} d^4x\,\sqrt{-g}(\mathbb{Q} + \lambda_{\alpha}{}^{\mu\beta\nu}R^{\alpha}{}_{\mu\beta\nu} + \delta_{\alpha}{}^{\mu\nu}T^{\alpha}{}_{\mu\nu})\,, \tag{3.88}
$$

where the Lagrange multiplier $\delta_{\alpha}{}^{\mu\nu}$ is antisymmetric with respect to the last two indices and $\lambda_{\alpha}{}^{\mu\beta\nu}$ is antisymmetry with respect to the first and second two indices.

In analogy to the superpotential $S^{\alpha\mu\nu}$ introduced for teleparallel general relativity, we introduce the *non-metricity conjugate* $P^{\alpha}{}_{\mu\nu}$, such that $\mathbb{Q} = Q_{\alpha}{}^{\mu\nu}P^{\alpha}{}_{\mu\nu}$, as follows

$$
P^{\alpha}{}_{\mu\nu} = \frac{1}{4}\bigg( - Q^{\alpha}{}_{\mu\nu} + Q_{\mu}{}^{\alpha}{}_{\nu} + Q_{\nu}{}^{\alpha}{}_{\mu} + Q^{\alpha\lambda}{}_{\lambda}g_{\mu\nu} + Q_{\lambda}{}^{\alpha\lambda}g_{\mu\nu} + \frac{1}{2}\Big(\delta^{\alpha}_{\mu}Q_{\nu}{}^{\lambda}{}_{\lambda} + \delta^{\alpha}_{\nu}Q_{\mu}{}^{\lambda}{}_{\lambda}\Big)\bigg)\,. \tag{3.89}
$$

The variation of the symmetric teleparallel action with respect to the metric leads to the following field equations

$$\frac{2}{\sqrt{-g}}\nabla_\alpha(\sqrt{-g}P^\alpha{}_{\mu\nu}) - q_{\mu\nu} - \mathbb{Q}g_{\mu\nu} = 0\,, \tag{3.90}$$

where

$$\begin{aligned} q_{\mu\nu} = &-\frac{1}{4}(2Q_{\alpha\beta\mu}Q^{\alpha\beta}{}_\nu - Q_{\mu\alpha\beta}Q_\nu{}^{\alpha\beta}) + \frac{1}{2}Q_{\alpha\beta\mu}Q^{\beta\alpha}{}_\nu \\ &+ \frac{1}{4}(2Q_\alpha{}^\lambda{}_\lambda Q^\alpha{}_{\mu\nu} - Q_\mu{}^\lambda{}_\lambda Q_\nu{}^\sigma{}_\sigma) + \frac{1}{2}Q_{\lambda\alpha}{}^\lambda Q^\alpha{}_{\mu\nu}\,. \end{aligned} \tag{3.91}$$

Note that also in this case the Lagrange multipliers do not enter into the field equations, therefore we can ignore the field equation for the Lagrange multiplier that one would get varying the action with respect to the connection. The field equations given in eq. (3.90) can be verified using the Riemann tensor of eq. (3.81) to compute the Einstein field equations $G_{\mu\nu} = 0$ as we did for teleparallel general relativity in section 3.3.5.

# 4 Entropy of Schwarzschild black holes

## 4.1 Overview

We use the Feynman gravitational path integral to relate the partition function to the Euclidean action for a Schwarzschild black hole. As we are going to see, the Euclidean action is completely determined by the boundary term. First, following S. Hawking [9] approach, we use the GHY boundary term to compute the action. Second, following N. Oshita and Y. Wu recent publication [20], we repeat the calculation using the teleparallel action and finally, we approach the problem using the symmetric teleparallel action. All methods deliver the same result which we use to calculate the entropy of Schwarzschild black holes, as S. Hawking first did in 1977 [9].

## 4.2 The gravitational path integral

In quantum mechanics, for a system with Hamiltonian $H$ and temperature $T = \beta^{-1}$, the partition function is

$$Z = \operatorname{Tr} e^{-\beta H} = \sum_n \langle g_n, \phi_n | e^{-\beta H} | g_n, \phi_n \rangle \,, \tag{4.1}$$

where $g$ and $\phi$ are gravitational and matter fields. We can relate this quantity to the time evolution operator $e^{-itH}$ by the Euclidean analytic coordinate transformation $t = -i\tau$, known as Wick rotation, where we identify the period of the Euclidean time $\tau$ to be $\beta$. The partition function can then be expressed as a path integral over all matter and gravitational fields as follows

$$Z = \int D[g] D[\phi] e^{-S_E(g,\phi)} \tag{4.2}$$

where $S_E(g, \phi) = -iS(g, \phi)$ is the Euclidean action and $S(g, \phi)$ is the gravitational action. This path integral is taken over all positive-definite metrics $g$ whose boundary is

$$\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2} \,, \tag{4.3}$$

where we consider the spacetime manifold $\mathcal{M}$ to be foliated by spacelike hypersurfaces $\Sigma_t = \mathcal{S}^3(r)$, a three-sphere of radius $r$ bounded by the closed two-surface $\mathcal{S}^2(r_0)$. The manifold itself is bounded by $\Sigma_{t_1}$, $\Sigma_{t_2}$ (where $t_1 = 0$, $t_2 = -i\beta$) and $\mathcal{B}(r_0) = [t_1, t_2] \times \mathcal{S}^2(r_0)$ the union of all two-surfaces $\mathcal{S}^2(r_0)$. The minus sign in front of $\Sigma_{t_1}$ serves to remember us that the normal vector to the hypersurface $\Sigma_{t_1}$ must point outwards $\mathcal{M}$. Figure 4.1 shows what we just said in a drawing.

The dominant contributions to the partition function $Z$ are given by those metrics $g$ and matter fields $\phi$ which are near the background fields $g_0$ and $\phi_0$, solutions of the classical field equations with the given periodicity and boundary conditions. Expanding the action in a Taylor series around the background fields, we obtain

$$S_E(g, \phi) \simeq S_E(g_0, \phi_0) + S^{(1)}(\bar{g}, \bar{\phi}) + ... \tag{4.4}$$

where $g = g_0 + \bar{g}$, $\phi = \phi_0 + \bar{\phi}$ and $S^{(1)}$ is quadratic in the perturbations $\bar{g}$ and $\bar{\phi}$. Then, inserting eq. (4.4) in eq. (4.2) and taking the natural logarithm, we get

$$\ln Z \simeq -S_E(g_0, \phi_0) + \ln \left( \int D[g] D[\phi] e^{-S^{(1)}(\bar{g},\bar{\phi})} \right) . \tag{4.5}$$

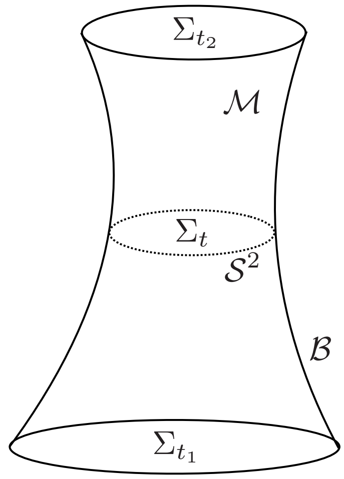


Figure 4.1: The manifold $\mathcal{M}$ is bounded by the two foliations $\Sigma_{t_1}$, $\Sigma_{t_2}$ and $\mathcal{B}$ the union of all two-surfaces $\mathcal{S}^2$.

As explained by Hawking [9], the first term of eq. (4.5) is the contribution of the background fields to the partition function, while the second term is the thermal contribution of thermal gravitons and matter quanta on the background geometry which from now on we will neglect. Furthermore, we will further simplify our calculations working in vacuum, i.e. we will neglect any matter field $\phi$.

## 4.3 The Euclidean section of the Schwarzschild solution

The simplest non-trivial solution of the vacuum Einstein field equations is the Schwarzschild solution, given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \,, \tag{4.6}$$

where $f(r) = \left(1 - \frac{2M}{r}\right)$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The Euclidean section of the Schwarzschild solution is obtained with the coordinate transformation $t = -i\tau$, known as Wick rotation, as follows

$$ds^2 = f(r)d\tau^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \,, \tag{4.7}$$

which makes the metric positive definite for $r > 2M$. The Euclidean Schwarzschild metric is asymptotically flat because $\lim_{r\to\infty} f(r) = 1$ and has a periodicity of $\beta = \frac{1}{T}$ in Euclidean time $\tau$. The singularity at $r = 2M$ is only apparent and is due to the choice of coordinate. To see this, we perform the transformation $x = rf(r)$ with $dx = dr$ and we obtain

$$ds^2 = \frac{x}{r}d\tau^2 + \frac{r}{x}dx^2 + r^2 d\Omega^2 \,, \tag{4.8}$$

then with $\chi^2 = x$ and $2\chi\, d\chi = dx$, we get

$$ds^2 = 4r\frac{\chi^2}{4r^2}d\tau^2 + 4rd\chi^2 + r^2 d\Omega^2 \,, \tag{4.9}$$

which is regular at $\chi = 0$, i.e. $r = 2M$. We now identify the two-dimensional geometry of the first two terms of the metric as flat space in polar coordinate, i.e. $ds^2 = d\chi + \chi^2 d\psi$ where $\psi$ has periodicity $2\pi$. From a simple coordinate transformation $\psi = \frac{\tau}{2r}$ we can determine the right periodicity of the Euclidean time to be $\beta = 4\pi r$. Note that the rescaling factor $4r$ does not affect the periodicity of $\tau$. On the horizon, i.e. $r = 2M$, we get

$$\beta = 4\pi r = 8\pi M \;\Rightarrow T_H = \beta^{-1} = \frac{1}{8\pi M}\,, \tag{4.10}$$

which is the *Hawking temperature* for the Schwarzschild black hole.

To summarize, we have shown that the Euclidean section of the Schwarzschild solution is defined for $\chi \geq 0$, $0 \leq \tau \leq 8\pi M$. Here the metric is positive definite, asymptotically flat and nonsingular.

To calculate the Euclidean action for the Schwarzschild metric we will need the induced metric on the hypersurface $\Sigma_{t_i} = \mathcal{S}^3(r)$, a three-sphere of radius $r$, and the three-cylinder $\mathcal{B}(r_0) = [t_1, t_2] \times \mathcal{S}^2(r_0)$, where $\mathcal{S}^2(r_0)$ is a two-surface of radius $r_0$. Knowing that the hypersurface $\Sigma_{t_i}$ is parametrized by $\Phi = t - t_i$ and the three-cylinder $\mathcal{B}$ by $\Psi = r - r_0$, we can derive the unit normal using eq. (2.6). Furthermore, once we know the unit normal we calculate the induced metric using eq. (2.9). Remembering that the Schwarzschild metric $g_{\mu\nu}$ is given in eq. (4.6), we compute these quantities as it follows.

The normal vector to the spacelike, i.e. $\varepsilon = -1$, hypersurface $\Sigma_{t_i}$ is

$$n_\mu = \frac{\varepsilon \partial_\mu \Phi}{|g^{\alpha\beta}\partial_\alpha \Phi \partial_\beta \Phi|} = -\sqrt{f(r)}(1,0,0,0)\,, \tag{4.11}$$

$$n^\mu = -\frac{1}{\sqrt{f(r)}}(1,0,0,0)\,, \tag{4.12}$$

the induced metric on $\Sigma_{t_i}$ is

$$h_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu = \mathrm{diag}(0, f(r)^{-1}, r^2, r^2 \sin^2\theta)\,, \tag{4.13}$$

and the square root of the determinant is $\sqrt{h} = f(r)^{-\frac{1}{2}} r^2 \sin\theta$. Furthermore, we indicate with $y^i$, $i = 1, 2, 3$, the coordinates on $\Sigma_{t_i}$.

The normal vector to the timelike, i.e. $\varepsilon = 1$, three-cylinder $\mathcal{B}(r_0)$ is

$$m_\mu = \frac{\varepsilon \partial_\mu \Psi}{|g^{\alpha\beta}\partial_\alpha \Psi \partial_\beta \Psi|} = \frac{1}{\sqrt{f(r)}}(0,1,0,0)\,, \tag{4.14}$$

$$m^\mu = \sqrt{f(r)}(0,1,0,0)\,, \tag{4.15}$$

the induced metric on $\mathcal{B}(r_0)$ is

$$\gamma_{\mu\nu} = g_{\mu\nu} - \varepsilon m_\mu m_\nu = \mathrm{diag}(-f(r_0), 0, r_0^2, r_0^2 \sin^2\theta)\,, \tag{4.16}$$

and the square root of the determinant is $\sqrt{-\gamma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta$. Furthermore, we indicate with $z^i$, $i = 0, 2, 3$ the coordinates on $\mathcal{B}(r_0)$.

Finally, the induced metric on the timelike, i.e. $\varepsilon = 1$, two-surface $\mathcal{S}^2(r_0)$ is

$$\sigma_{\mu\nu} = h_{\mu\nu} - \varepsilon m_\mu m_\nu = \mathrm{diag}(0, 0, r_0^2, r_0^2 \sin^2\theta)\,, \tag{4.17}$$

with $\sqrt{\sigma} = r_0^2 \sin\theta$ and we indicate with $w^i$, $i = 2, 3$ the coordinates on $\mathcal{S}^2(r_0)$. Note that $\sqrt{-\gamma} = N\sqrt{\sigma}$ where $N = f(r_0)^{1/2}$ is known as the lapse function in the ADM formalism.

## 4.4 Gibbons-Hawking-York approach

Here we will follow the approach used by S. Hawking in 1977 [9] to derive the Euclidean action of a Schwarzschild black hole using the $S_{GHY}$ counter term. Since in Hawking's paper the calculations are not carried out, this is a good exercise to test the knowledge acquired so far.

Before proceeding with the derivation, for convenience, we write down the Einstein-Hilbert action corrected with the GHY counter term obtained in section 2.2. Since we are interested in solutions of the Einstein field equations that are asymptotically flat at infinite, we chose the constant of integration $S_0$ in eq. (2.38) to be the one calculated in section 2.2.4. The action takes therefore the following form

$$S = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R + \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^3y \, \varepsilon \sqrt{|h|} (K - K_0) \tag{4.18}$$

where the surface integral is carried out on $\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2}$ and $K_0 = \frac{2}{r_0}$ is the trace of the extrinsic curvature tensor for asymptotically flat spacetime computed in eq. (2.41).

We now calculate the Euclidean action $S_E = -iS$ for the Schwarzschild metric. Because the Schwarzschild metric is a solution of the vacuum Einstein field equations, the Ricci scalar vanishes and therefore $S_{EH} = 0$, i.e. the Euclidean action is completely determined by the $S_{GHY}$ boundary term. To compute the action, we need to compute the extrinsic curvature on the boundary $\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2}$ where $t_1 = 0$ and $t_2 = -i\beta$.

The extrinsic curvature $K$ on the two hypersurfaces $\Sigma_{t_1}$, $\Sigma_{t_2}$ vanishes. Using the unit normal $n^\mu$ to $\Sigma_t$ given in eq. (4.12), we get

$$K = \nabla_\mu n^\mu = \partial_\mu n^\mu + \Gamma^\mu_{\mu\lambda} n^\lambda = \Gamma^\mu_{\mu t} n^t = 0 \tag{4.19}$$

because the Christoffel symbol $\Gamma^\mu_{\mu t}$ vanishes.

While the three-cylinder $\mathcal{B}(r_0)$ has a non zero extrinsic curvature. Using the unit normal $m^\mu$ to $\mathcal{B}(r_0)$ computed in eq. (4.15) and the induced metric $\gamma_{\mu\nu}$ on $\mathcal{B}(r_0)$ given in eq. (4.16), we get

$$\begin{aligned} K &= \nabla_\mu m^\mu = \partial_\mu m^\mu + \Gamma^\mu_{\mu\lambda} m^\lambda = \Gamma^\mu_{\mu r} m^r = f(r)^{\frac{1}{2}} \, \frac{1}{2} \gamma^{\mu\nu} \partial_r \gamma_{\mu\nu} \bigg|_{r=r_0} = \\ &= \frac{1}{2} f(r)^{\frac{1}{2}} (\gamma^{tt} \partial_r \gamma_{tt} + \gamma^{\phi\phi} \partial_r \gamma_{\phi\phi} + \gamma^{\theta\theta} \partial_r \gamma_{\theta\theta}) \bigg|_{r=r_0} = \\ &= \frac{1}{2} f(r_0)^{\frac{1}{2}} \left( f(r_0)^{-1} \frac{2M}{r_0^2} + \frac{2}{r_0} + \frac{2}{r_0} \right) = \frac{2}{r_0} f(r_0)^{\frac{1}{2}} + f(r_0)^{-\frac{1}{2}} \frac{M}{r_0^2} \, . \end{aligned} \tag{4.20}$$

Since the two hypersurfaces $\Sigma_{t_1}$, $\Sigma_{t_2}$ have a vanishing extrinsic curvature, the only contribution to the Euclidean action is the one of the boundary $\mathcal{B}(r_0)$

$$S_E \equiv -iS = \frac{-i}{8\pi} \int_{\mathcal{B}(r_0)} d^3z \sqrt{-\gamma} \, (K - K_0) = \frac{-i}{8\pi} \int_0^{-i\beta} dt \int_{\mathcal{S}^2(r_0)} d^2w \sqrt{-\gamma} \, (K - K_0) \, . \tag{4.21}$$

Therefore, using $\sqrt{-\gamma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta$ obtained with eq. (4.16) together with the extrinsic curvature K of eq. (4.20) and $K_0 = \frac{2}{r_0}$ , we calculate the Euclidean action as follows

$$
\begin{aligned}
S_E &= -\frac{i}{8\pi}\int_0^{-i\beta} dt \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{-\gamma}\,(K - K_0) = \\
&= -\frac{\beta}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta \left(\frac{2}{r_0} f(r_0)^{\frac{1}{2}} + f(r_0)^{-\frac{1}{2}}\frac{M}{r_0^2} - \frac{2}{r_0}\right) = \\
&= \beta\left(\frac{3}{2}M - r_0 + f(r_0)^{\frac{1}{2}} r_0\right) \simeq \beta\left(\frac{3}{2}M - r_0 + \left(1 - \frac{M}{r_0} + \mathcal{O}\!\left(\frac{M^2}{r_0^2}\right)\right) r_0\right) = \\
&= \frac{\beta}{2}M + \mathcal{O}\!\left(\frac{1}{r_0}\right) \underset{r_0\to\infty}{\longrightarrow} \frac{\beta}{2}M\,. 
\end{aligned} \tag{4.22}
$$

where we used the Taylor series of $f(r_0)^{1/2} = (1 - \frac{2M}{r_0})^{1/2} \simeq 1 - \frac{M}{r_0} + \mathcal{O}\left(\frac{M^2}{r_0^2}\right)$. Remembering from the previous section that the period of the Euclidean time is $\beta = 8\pi M$, we get the following result for the Euclidean action of a Schwarzschild black hole

$$
S_E = \frac{\beta}{2}M = \frac{\beta^2}{16\pi} = 4\pi M^2\,. \tag{4.23}
$$

## 4.5 Teleparallel approach

To my knowledge, N. Oshita and Y. Wu [20] were the first to publish in August 2017 the calculation of the Euclidean action for a Schwarzschild black hole using the teleparallel action of general relativity. As an exercise, adopting our notation, we will go through the calculations in more details.

In section 3.3.3 we showed that the torsion induced surface term of the teleparallel action plays the same role as the GHY boundary term. Since the two approaches are equivalent, we expect to recover the same result as in the GHY approach when computing the Euclidean action with the teleparallel action. Choosing the constant of integration $S_0$ in eq. (3.43) to be regularized by a reference backgrond defined by the flat spacetime as we did for the GHY term, the teleparallel action given in eq. (3.25) becomes

$$
\begin{aligned}
S_T &= \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-g}R + \frac{1}{8\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\nabla_\mu\left(T^{\alpha\mu}{}_\alpha - (T^{\alpha\mu}{}_\alpha)_0\right) = \\
&= \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3y\,\varepsilon\sqrt{|h|}n_\mu\left(T^{\alpha\mu}{}_\alpha - (T^{\alpha\mu}{}_\alpha)_0\right)\,,
\end{aligned} \tag{4.24}
$$

where we used once again Gauss' theorem to compute the second term on the boundary $\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2}$. To compute the Euclidean action we will set $t_1 = 0$ and $t_2 = -i\beta$.

Since the Schwarzschild metric is a solution of the vacuum Einstein fields equations, the Ricci scalar vanishes, leaving the Euclidean action completely determined by the torsion induced term. Defining $T_\mu := T^\alpha{}_{\mu\alpha} - (T^\alpha{}_{\mu\alpha})_0$ for convenience, the Euclidean action is

$$
\begin{aligned}
S_E = -iS_T &= -\frac{i}{8\pi}\left(-\int_{-\Sigma_{t_1}} d^3y\,\sqrt{h}\,n^\mu T_\mu + \int_{\mathcal{B}(r_0)} d^3z\,\sqrt{-\gamma}\,m^\mu T_\mu - \int_{\Sigma_{t_2}} d^3y\,\sqrt{h}\,n^\mu T_\mu\right) = \\
&= -\frac{i}{8\pi}\int_0^{-i\beta} dt \int_{\mathcal{S}^2(r_0)} d^2w\,\sqrt{-\gamma}\,m^r\left(T^\alpha{}_{r\alpha} - (T^\alpha{}_{r\alpha})_0\right)\big|_{r=r_0}\,,
\end{aligned} \tag{4.25}
$$

where we used that the two integrals on the hypersurfaces $\Sigma_{t_1}$ and $\Sigma_{t_2}$ with $t_1 = 0$, $t_2 = -i\beta$ cancel each other out[10].

The tetrad field of the Schwarzschild metric given in eq. (4.6) can be read out, using eq. (2.90), by rewriting the metric in the following form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}e_\mu{}^a e_\nu{}^b dx^\mu dx^\nu$ where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. We then obtain

$$e_\mu{}^a = \left(f(r)^{\frac{1}{2}}, f(r)^{-\frac{1}{2}}, r, r\sin\theta\right) , \tag{4.26}$$

$$e^\mu{}_a = \left(f(r)^{-\frac{1}{2}}, f(r)^{\frac{1}{2}}, r^{-1}, r^{-1}\sin^{-1}\theta\right) , \tag{4.27}$$

where $f(r) = \left(1 - \frac{2M}{r}\right)$. We now comoute the torsion tensor $T^\alpha{}_{r\alpha}$. Using eq. (3.13), i.e. $T^\lambda{}_{\mu\nu} = e^\lambda{}_a(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a)$, we compute

$$T^t{}_{rt} = e^t{}_a(\partial_r e_t{}^a - \partial_t e_r{}^a) = \frac{1}{2}\frac{f'(r)}{f(r)} , \tag{4.28}$$

$$T^r{}_{rr} = e^r{}_a(\partial_r e_r{}^a - \partial_r e_r{}^a) = 0 , \tag{4.29}$$

$$T^\phi{}_{r\phi} = e^\phi{}_a(\partial_r e_\phi{}^a - \partial_\phi e_r{}^a) = \frac{1}{r} , \tag{4.30}$$

$$T^\theta{}_{r\theta} = e^\theta{}_a(\partial_r e_\theta{}^a - \partial_\theta e_r{}^a) = \frac{1}{r} , \tag{4.31}$$

and we obtain

$$T^\alpha{}_{r\alpha} = T^t{}_{rt} + T^r{}_{rr} + T^\phi{}_{r\phi} + T^\theta{}_{r\theta} = \frac{1}{2}\frac{f'(r)}{f(r)} + \frac{2}{r} . \tag{4.32}$$

Using $m^\mu$ given in eq. (4.15) and $\sqrt{-\gamma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta$ obtained with eq. (4.16), we compute the following quantity on the boundary $\mathcal{B}(r_0)$

$$\sqrt{-\gamma}\, m^r T^\alpha{}_{r\alpha}|_{r=r_0} = f(r_0) r_0^2 \sin\theta \left(\frac{1}{2}\frac{f'(r_0)}{f(r_0)} + \frac{2}{r_0}\right) = \sin\theta\,(-3M + 2r_0) . \tag{4.33}$$

The tetrad field of the asymptotically flat spacetime is

$$e_\mu{}^a = (1, 1, r, r\sin\theta) , \tag{4.34}$$

$$e^\mu{}_a = \left(1, 1, r^{-1}, r^{-1}\sin^{-1}\theta\right) , \tag{4.35}$$

which we use to compute the torsion term $(T^\alpha{}_{r\alpha})_0$ and obtain

$$(T^\alpha{}_{r\alpha})_0 = T^t{}_{rt} + T^r{}_{rr} + T^\phi{}_{r\phi} + T^\theta{}_{r\theta} = \frac{2}{r} . \tag{4.36}$$

[10] They cancel each other out because the argument of both integrals is the same and is independt of $t$. We use $e_\mu{}^a$ given in eq. (4.26) to compute the argument of the integral following the same procedure done for the torsion of the three-cylinder $\mathcal{B}(r_0)$: $n^\mu T^\alpha{}_{\mu\alpha} - (n^\mu T^\alpha{}_{\mu\alpha})_0 = n^t T^r{}_{tr} = -n^t e^r{}_a \partial_r e_t{}^a = \frac{f'(r)}{2f(r)^{3/2}} =: F(r)$ which does not depend on $t$. One may see as follows that the two integrals cancel each other out $\int_{-\Sigma_{t_1}} F(r) + \int_{\Sigma_{t_2}} F(r) = -\int_{\Sigma_t} F(r) + \int_{\Sigma_t} F(r) = 0$.

The normal vector of the asymptotically flat spacetime to the three cylinder $\mathcal{B}(r_0)$ is obtained using the metric of asymptotically flat spacetime and is $m^\mu = (0, 1, 0, 0)$. Therefore, using again $\sqrt{-\gamma} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta$, we compute the following quantity on the boundary $\mathcal{B}(r_0)$

$$\sqrt{-\gamma}\, m^r (T^\nu{}_{r\nu})_0|_{r=r_0} = f(r_0)^{\frac{1}{2}} r_0^2 \sin\theta \frac{2}{r_0} \simeq \sin\theta\, 2r_0 \left(1 - \frac{M}{r_0} + \mathcal{O}\left(\frac{M^2}{r_0^2}\right)\right) = \\ = \sin\theta \left(2r_0 - 2M + \mathcal{O}\left(\frac{1}{r_0}\right)\right) , \tag{4.37}$$

where we used the Taylor series of $f(r_0)^{1/2} = (1 - \frac{2M}{r_0})^{1/2} \simeq 1 - \frac{M}{r_0} + \mathcal{O}\Big(\frac{M^2}{r_0^2}\Big)$.

Plugging in the results of eq. (4.33) and eq. (4.37) in eq. (4.41), we obtain

$$S_E = -\frac{i}{8\pi} \int_0^{-i\beta} dt \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{-\gamma}\, m^r \left(T^\alpha{}_{r\alpha} - (T^\alpha{}_{r\alpha})_0\right)|_{r=r_0} = \\ = -\frac{\beta}{8\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \left(-M + \mathcal{O}\left(\frac{1}{r_0}\right)\right) = \\ = \frac{\beta}{2} M + \mathcal{O}\left(\frac{1}{r_0}\right) \underset{r_0\to\infty}{\to} \frac{\beta}{2} M . \tag{4.38}$$

Recalling that the period of the Euclidean time is $\beta = 8\pi M$, we get the following result for the Euclidean action of a Schwarzschild black hole

$$S_E = \frac{\beta}{2} M = \frac{\beta^2}{16\pi} = 4\pi M^2 , \tag{4.39}$$

which agrees with the result obtained using the Gibbons-Hawking-York approach.

## 4.6 Symmetric teleparallel approach

Here we are going to calculate the Euclidean action for a Schwarzschild black hole using the symmetric teleparallel action of general relativity. To my knowledge, this is not been done in the literature. Inserting the result obtained in eq. (3.87) into the definition of the symmetric teleparallel action (3.77), we obtain

$$S_Q = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R + \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} \nabla_\mu (Q^{\mu\alpha}{}_\alpha - Q_\alpha{}^{\mu\alpha}) = \\ = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R + \frac{1}{16\pi} \int_{\partial\mathcal{M}} d^3y\, \varepsilon \sqrt{|h|} n_\mu (Q^{\mu\alpha}{}_\alpha - Q_\alpha{}^{\mu\alpha}) , \tag{4.40}$$

where in the second equality we used as usual the Gauss' theorem to calculate the second term on the boundary $\partial\mathcal{M} = (-\Sigma_{t_1}) \cup \mathcal{B} \cup \Sigma_{t_2}$ where to calculate the Euclidean action we will set $t_1 = 0$ and $t_2 = -i\beta$.

Since the Schwarzschild metric is a solution of the vacuum Einstein fields equations, the Ricci scalar vanishes, leaving the Euclidean action completely determined by the non-metricity term. As explained in section 3.4.1, we can choose the gauge condition such that the connection of the symmetric teleparallel action vanishes, i.e. $\widetilde{\Gamma}^\mu{}_{\alpha\beta} = 0$. In this gauge the non-metricity tensor takes the following form $Q_{\mu\alpha\beta} = \widetilde{\nabla}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta}$ and it implies that the term $Q^{\mu\alpha}{}_\alpha$ in eq. (4.40) vanishes because $Q^{\mu\alpha}{}_\alpha = \partial^\mu g^\alpha{}_\alpha = \partial^\mu \delta^\alpha_\alpha = 0$. Noticing that on the hypersurfaces $\Sigma_{t_1}$ and $\Sigma_{t_2}$ it holds $n_\mu Q_\alpha{}^{\mu\alpha} = n_t Q_\alpha{}^{t\alpha} = n_t \partial_t g^{tt} = 0$, we compute the Euclidean action as follows

$$\begin{aligned}
S_E = -iS_Q = -\frac{i}{16\pi}\left(\int_{\Sigma_{t_1}} d^3y\,\sqrt{h}\,n_\mu Q_\alpha{}^{\mu\alpha} - \int_{\mathcal{B}(r_0)} d^3z\,\sqrt{-\gamma}\,m_\mu Q_\alpha{}^{\mu\alpha} + \int_{\Sigma_{t_2}} d^3y\,\sqrt{h}\,n_\mu Q_\alpha{}^{\mu\alpha}\right) = \\
= \frac{i}{16\pi}\int_0^{-i\beta} dt \int_{\mathcal{S}^2(r_0)} d^2w\,\sqrt{-\gamma}\,m_r Q_\alpha{}^{r\alpha}|_{r=r_0} = \\
= \frac{i}{16\pi}\int_0^{-i\beta} dt \int_{\mathcal{S}^2(r_0)} d^2w\,\sqrt{-\gamma}\,m_r \partial_r g^{rr}|_{r=r_0} = \\
= \frac{\beta}{16\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{f(r_0)} r_0^2 \sin\theta \frac{1}{\sqrt{f(r_0)}}\partial_r\left(1 - \frac{2M}{r}\right)\bigg|_{r=r_0} = \\
= \frac{\beta}{2}M\,. \qquad (4.41)
\end{aligned}$$

Note that for the symmetric teleparallel action the constant of integration $S_0$, which makes the action physical, is trivial because the boundary term is completely determined by the non-metricity which does not diverge at infinite due to the choice of coordinates.

Recalling that the period of the Euclidean time is $\beta = 8\pi M$, we get the following result for the Euclidean action of a Schwarzschild black hole

$$S_E = \frac{\beta}{2}M = \frac{\beta^2}{16\pi} = 4\pi M^2\,, \qquad (4.42)$$

which agrees with the results obtained using the Gibbons-Hawking-York and teleparallel approaches.

## 4.7 The entropy of a Schwarzschild black holes

We now use the first leading order approximation of the partition function given in eq. (4.5) to compute the partition function of a Schwarzschild black hole, i.e.

$$\ln Z \simeq -S_E(g_0) = -\frac{\beta^2}{16\pi}\,, \qquad (4.43)$$

which we use to calculate the entropy on the event horizon. From thermodynamics we know that the energy of the system relates to the partition function as

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\beta}{8\pi} = M \qquad (4.44)$$

while the Bekenstein-Hawking entropy is obtain from $\ln Z = S - \beta E$ as follows

$$S = \ln Z + \beta E = (1 - \beta\partial_\beta)\ln Z = \frac{\beta^2}{16\pi} = 4\pi M^2 = \frac{A}{4} \qquad (4.45)$$

where in the last equality we substituted the surface area of the Schwarzschild black hole event horizon, i.e. at the Schwarzschild radius r=2M one obtains $A = 4\pi r^2 = 16\pi M^2$.

# 5 Conclusions

We solved the variational problem of the Einstein-Hilbert action for spacetime manifolds with boundaries. This required us to correct the Einstein-Hilbert action with a counter-term. For spacelike and timelike boundaries the counter-term is the GHY boundary term, while for null-like boundaries the counter term is the one found by K. Parattu *et al.* [8]. We want to draw the attention of the reader to the fact that although this problem was solved for non-null boundaries more than forty years ago, it is surprising that not until recently somebody has found a solution for null-like boundaries. Another important point on which we want to draw the attention to, is that the tetrad formulation of general relativity allowed us to obtain both boundary terms with much less efforts.

The study of boundary terms brought us to explore alternative spacetime geometries of general relativity. We studied the teleparallel action of Weitzenböck spacetime and the symmetric teleparallel acton of symmetric teleparallel spacetime showing that these alternative geometries are equivalent to the Einstein-Hilbert action of Riemann spacetime up to a divergence term, i.e. the boundary term. While these three geometries use different properties of the spacetime manifold to describe the effects of gravity, they are equivalent up to the boundary term. We compared these three alternative geometries computing the Euclidean action for a Schwarzschild black hole. We used this result to compute the entropy of the black hole. All the three approaches delivered the same result.

We worked out the entropy of a Schwarzschild black hole just an example on how one may find an astrophysical application to compare these different formulations of general relativity. We hope to have aroused the reader's curiosity and to have made him aware of this ambiguity in the description of spacetime manifolds in general relativity. An interesting outlook of this work could be to further investigate the differences between these three formulations studying for example the coupling of matter to gravity. It is well known that Dirac fermions do not couple with the non-metricity but do couple with the contorsion part of the affine connection, one may expect that the different boundary terms play different roles in the coupling. Furthermore, if fermions have a Lagrangian that goes beyond the Dirac Lagrangian, then even the non-metricity could play an important role in the coupling.

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