An Update on the Impact of Schiff's Conjecture on Tests of the Einstein Equivalence Principle Today

Damian Moosbrugger
ETH Zürich
damianmo@ethz.ch

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Supervisors:
Prof. Dr. Philippe Jetzer[1] and Prof. Dr. Niklas Beisert[2]

Abstract: We present here an update of the confirmation of the three pillars of the Einstein Equivalence Principle, namely the Weak Equivalence Principle, Local Lorentz Invariance and Local Position Invariance, by taking the first results of the MICROSCOPE mission into account. The MICROSCOPE mission is an Eötvös experiment in space whose findings effected the reduction of the upper limit of the "Eötvös ratio" η to $|\eta(\mathrm{Ti},\mathrm{Pt})| \leq 1.3 \times 10^{-14}$ in December 2017 [44]. In a first step, the most stringent limits on the strength of violation of these three pillars are calculated under the assumption that they, as well as the experiments which test them, have been conducted wholly independent of each other. In a second step, a quantitative relationship between the various violation parameters is derived and applied to recent and possible future results. This step assumes the validity of Schiff's conjecture, which goes back to a paper by Leonard I. Schiff [41]. It is shown that under these assumptions, the new result as obtained by the MICROSCOPE mission yields the most stringent constraints, not only on violations of the Weak Equivalence Principle, but also on violations of Local Lorentz and Position Invariance. In some cases, the considerable strength of the constraints indicate that current or future experiments of a different kind will not be able to yield such significant results. For example the upper bound on the strength of violation of Local Position Invariance by the hyperfine interaction is already set to $|\alpha^{\mathrm{HF}}| \leq 4.1 \times 10^{-9}$ by the MICROSCOPE mission, whereas the planned ACES mission, a gravitational redshift experiment, aims to get to parts in 10^6 and should therefore not be able to detect any deviation. The advantages for future confirmation, which a performance of experiments testing possible violations of the Einstein Equivalence Principle by the strong or the weak, instead of the electromagnetic interaction, are expected to have, are presented in this thesis, too.

[1]Physik-Institut, University of Zurich, Y36-K82, Winterthurerstr. 190, 8057 Zurich, Switzerland
[2]Institut für Theoretische Physik, ETH Zurich, HIT K 31.8, Wolfgang-Pauli-Str. 27, 8093 Zurich, Switzerland

Contents

1 Introduction

The different assumptions of the Principle of Equivalence (EP), which were the starting points for the formulation of the general theory of relativity (GR), first written down by Albert Einstein in his famous work "Grundlagen der allgemeinen Relativitätstheorie"[3] [16] in 1916, have been confirmed by a broad range of experiments to a very accurate level and their further confirmation is still an active field of research. For example in December 2017, the most stringent limit of a possible violation of the Universality of Free Fall (UFF), also referred to as Weak Equivalence Principle (WEP), found to date was set by the "MICRO-Satellite á traînée Compensée pour l'Observation du Principe d'Equivalence" (MICROSCOPE) mission to parts in 10^{14} and aims to get to parts in 10^{15} [44]. Such tests of the WEP have even been called an "*Experimentum Crucis* of modern physics" (page 2) [30], because a deviation from the WEP, as well as its more stringent confirmation will either way have a considerable impact on various areas in theoretical physics such as quantum gravity. For example string theory predicts a violation of the WEP at some level.

The three pillars of the Einstein Equivalence Principle (EEP), which consist of the WEP, Local Lorentz Invariance (LLI) and Local Position Invariance (LPI), that is Lorentz and position invariance in locally free-falling frames, are tested in experiments of a different type. The most important ones are the Eötvös experiments (comparison of the acceleration of different test bodies) for the WEP, Hughes and Drever experiments (splitting of energy levels of atoms due to preferred frame effects) for the LLI and gravitational redshift experiments (comparison of clocks in different gravitational potentials) for LPI. These three aspects of the EEP are usually tested and analyzed wholly independent of each other. There is, however, a conjecture which goes back to Leonard I. Schiff and his article published in 1960 "On Experimental Tests of the General Theory of Relativity" [41], which argues that the different experiments are not just independent confirmations of single aspects of the EEP, and thus the metric nature of gravitation. Rather, Schiff's conjecture states that there exists a quantitative relationship between possible deviations from the basic assumptions. Formulated differently, the confirmation of the WEP alone directly implies verification of the EEP. Tests of the WEP are therefore at the same time tests of LLI and LPI, too, and the other way around.

This thesis is structured as follows: In section 2 the different EP's and their impact on the theory of gravitation are introduced, focusing particularly on the EEP. In section 3 the most important tests which were or are going to be performed to confirm the EEP are outlined. The procedure in these two sections mainly adheres to the procedure demonstrated in the book of Clifford M. Will *Theory and Experiment in Gravitational Physics* [50] and his Living Review "The Confrontation between Relativity and Experiment". A "Gedankenexperiment"[4] based on the principle of energy conservation supporting Schiff's conjecture, which states that there is a quantitative relationship between the different tests of the EEP is presented in section 4. It mainly adheres to the procedures in the studies done in 1975 by Kenneth L. Nordtvedt Jr. [33] and in 1979 by Mark P. Haugan [19]. Approximative expressions for the amount of energy of different forms (electromagnetic, strong, weak etc.) of which the test bodies consist are presented in section 5. By using the relationship derived in section 4 and recent results from the experiments, the most stringent limits on any possible violation of the EEP found until today are calculated. In section 6 this is done by assuming that the three pillars are completely independent of each other, and in section 7 by use of the relationship arising from Schiff's conjecture. A short introduction to the $TH\epsilon\mu$ formalism, developed by Alan P. Lightman and David L. Lee, and a comparison of results received by using this framework with values obtained in the previous

[3]Translation from German to English: "Foundations of the General Theory of Relativity"
[4]Translation from German to English: "Thought experiment"

sections are made in section 8. The thesis closes with section 9 in which a conclusion on the current state of confirmation is made and suggestions for (possible) future research are outlined.

2 The Principle of Equivalence

"Quantitas Materiæest mensura ejusdem orta ex illius Densitate & Magnitudine conjunctim. [...] Hanc autem quantitatem sub nomine corporis vel Massæin sequentibus passim intelligo. Innotescit ea per corporis cujusq; pondus. Nam ponderi proportionalem esse reperi per experimenta pendulorum accuratissime instituta, uti posthac docebitur"[5]

- Sir Isaac Newton, *PhilosophiæNaturalis Principia Mathematica*, page 1 [28].

There exist a variety of different formulations and types of EP which contain different assumptions and are therefore referred to as "weaker" or "stronger". The convention in this thesis will follow the one summarized by Eolo Di Casola, Stefano Liberati and Sebastiano Sonego [8], where we will mainly focus on the Weak, the Einstein and the Strong Equivalence Principle.

2.1 The Weak Equivalence Principle

The WEP or UFF states that all test bodies with negligible self-gravity behave the same in a gravitational field, meaning they experience the same acceleration, independent of their internal structure or composition. This principle is such a cornerstone of gravitational physics that it was already tested and described by Galileo Galilei in his Discorsi [17] or by Sir Isaac Newton, who formulated it in the first definition in the opening paragraph of his Principia (see above). Hermann Bondi named the terms (matter and weight) inertial mass m_I and (passive) gravitational mass m_G[6] [5], so that the principle of equivalence can be written as

$$m_I = m_G. \tag{1}$$

A more precise definition of the WEP as given e.g. by Clifford M. Will is the following: "If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition" (page 22) [50]. An uncharged test body in this context means an electrically neutral body with negligible self-gravitational energy. There is a parameter σ to define when self-gravity is negligible [8]

$$\sigma = \frac{Gm}{c^2 r}, \tag{2}$$

where G is the gravitational constant, m is the mass of the test body (it does not matter if gravitational or inertial mass, since they are assumed to be equal), c is the speed of light and r is the size of the test body. For example an atom can be regarded as a test body, since it is uncharged and the self-gravitational energy is negligible as $\sigma \sim 10^{-43}$.

[5]Translation from Latin to English: "The quantity of matter is the measure of the same, arising from its density and bulk conjointly. [...] And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter."

[6]The passive gravitational mass denotes the mass that gets affected by gravity, in contrast to the active gravitational mass, which creates the gravitational field.

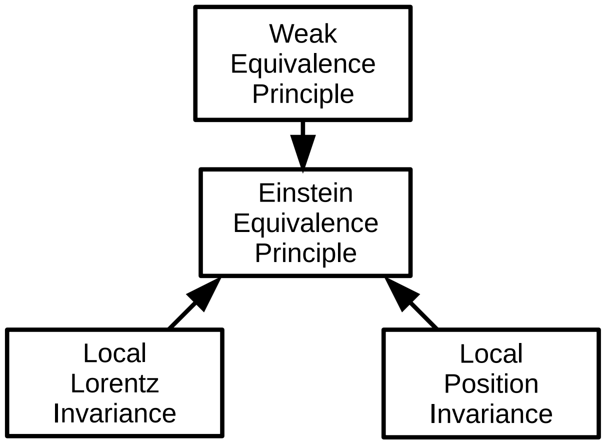

Figure 1: Illustration of the three pillars of the EEP. Before the conjecture of Leonard I. Schiff, they were regarded as three statements, completely independent of each other.

2.2 The Einstein Equivalence Principle

Albert Einstein stated, that if all the bodies fall with the same acceleration in an external gravitational field, then in a freely falling system, the mechanical laws will behave as if gravity were absent. He even added that not only mechanical laws, but all the laws of physics should behave this way. This principle was used by Einstein for the development of GR (see e.g.[14]) and still is its foundation.

The EEP consists of the following three assumptions, which may appear as completely independent statements in the first place.

1. The *Weak Equivalence Principle* (WEP), which was introduced above in the previous subsection, is valid and therefore all test bodies experience the same acceleration in a gravitational field. This assumption is essential for the existence of any local freely falling frames.

2. *Local Lorentz Invariance* (LLI): The outcome of any local[7] non-gravitational experiment is independent of the velocity of the freely falling reference frame in which it is performed.

3. *Local Position Invariance* (LPI): The outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed. The last two points therefore state that there do neither exist any preferred frame nor location effects. In Figure 1, a schematic representation of the so called three pillars of the EEP is given. It can be shown that (see e.g. [50]) if all the three pillars and therefore the EEP are valid, the theory of gravity must satisfy the postulates of metric theories of gravity and can therefore be called a metric theory. This contains the following conditions: First, that spacetime is endowed with a symmetric metric, which in the case of GR would be the metric tensor $g_{\mu\nu}$. Second, that the trajectories of freely falling test bodies are geodesics of that metric. A test body in a gravitational field described by the metric $g_{\mu\nu}(x)$, which may depend on coordinates x and therefore correspond to a curved

[7]A local test experiment in this context means that it is small, such that there exist no inhomogeneities in the gravitational potential. This definition was first used by Kip S. Thorne, David L. Lee and Alan P. Lightman [42].

spacetime, fulfills the following equation of motion

$$\frac{d^2x^\kappa}{d\tau^2} = -\Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \quad (3)$$

where τ and $\Gamma^\kappa_{\mu\nu}$ are the proper time and the Christoffel symbols, respectively, which are connected to the metric by the following expression (which can both be found in standard textbooks on GR)

$$\Gamma^\kappa_{\mu\nu} = \frac{g^{\kappa\lambda}}{2}\left(\frac{\partial g_{\lambda\nu}}{\partial x^\mu} + \frac{\partial g_{\lambda\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right). \quad (4)$$

The solution of the equation of motion of the test body leads to geodesics of $g_{\mu\nu}(x)$. The third and last postulate is, that in local freely falling reference frames, the non-gravitational laws of physics are those written in the language of special relativity. The EEP can therefore be regarded as the foundation of all curved spacetime or metric theories of gravity, which means that all non-gravitational fields couple in the same way to the gravitational field. If it is satisfied, the theory of gravitation must be a phenomenon of curved spacetime.

2.3 The Strong Equivalence Principle

In contrast to the EEP, the Strong Equivalence Principle (SEP) includes all types of test bodies, even gravitationally bound ones. Instead of the WEP, it therefore contains the Gravitational Weak Equivalence Principle (GWEP) as one of the three pillars, which states that all the test particles behave the same way in a gravitational field, independent of their internal structure or composition. It includes the WEP in the limit where $\sigma \to 0$. We see that the EEP can be treated as a special case of the SEP, when gravitational forces are negligible, and its validation is at the same time a validation of the EEP. Up to now, GR is the only known theory completely fulfilling the SEP[8]. As a consequence its violation would only falsify GR but not other metric theories.

3 Tests of the Einstein Equivalence Principle

The three parts of the EEP, namely the WEP, LLI and LPI, are usually tested independently of each other. However, we will see that if Schiff's conjecture is correct, certain experiments yield confirmations on more than one point of the EEP at the same time. In Figure 3 all the different experiments which will be discussed below as well as the statements they test are illustrated.

3.1 Tests of the Weak Equivalence Principle

The WEP can be tested by so called Eötvös experiments[9], in which the acceleration due to gravity of two different test bodies is compared. If the WEP is valid, every test body, independent of their internal structure or composition should experience the same acceleration in vacuum. A possible violation of WEP can be introduced, by allowing a certain form of energy A (e.g electromagnetic

[8]The only other theory known, which fulfills the SEP is the conformally flat scalar theory developed by Gunnar Nordstrm in 1913 [32]. However, it is experimentally ruled out since it predicts no deflection of light.

[9]Named after Baron Loránd Eötvös de Vásárosnamény (1848-1919), better known as Roland von Eötvös, and the experiments he performed. He was the inventor of the torsion balance, which led to great improvements in the validation of the WEP.

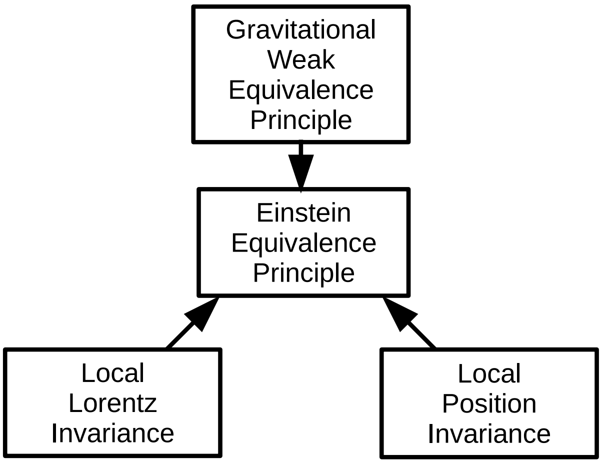

Figure 2: Illustration of the three pillars of the SEP. In contrast to the EEP which contains the WEP as a pillar, it assumes the GWEP. One could therefore say that the SEP is equivalent to the union of the EEP and GWEP.

or hyperfine energy) to contribute differently to the gravitational mass m_G than it does to the inertial mass m_I by

$$m_G = m_I + \sum_A \frac{\eta^A E^A}{c^2}, \tag{5}$$

where E^A is the amount of energy of form A and η^A is a dimensionless parameter which measures the strength of its violation of the WEP. The acceleration of a test body with gravitational mass m_G and inertial mass m_I under influence of a gravitational field is given by using Newton's law

$$a = \frac{m_G}{m_I} g = \left[1 + \sum_A \eta^A \left(\frac{E^A}{m_I c^2}\right)\right] g, \tag{6}$$

where g is the gravitational acceleration. For a test body X with a negligible amount of self-gravitational energy, its acceleration in a gravitational field can be written as

$$a_X = \left(1 + \sum_A \eta^A \zeta_X^A\right) g, \tag{7}$$

where the fractional energy contribution of interaction A to the total energy of a test body X is introduced by

$$\zeta_X^A = \left.\frac{E^A}{m_I c^2}\right|_X, \tag{8}$$

and will be frequently used in the following sections. To measure the violation of the WEP in experiments, the Eötvös ratio η, which is a measure of the difference in acceleration two test bodies X and Y experience in a gravitational field, is used. It is defined as

$$\eta(X,Y) \equiv \frac{2(a_X - a_Y)}{a_X + a_Y}. \tag{9}$$

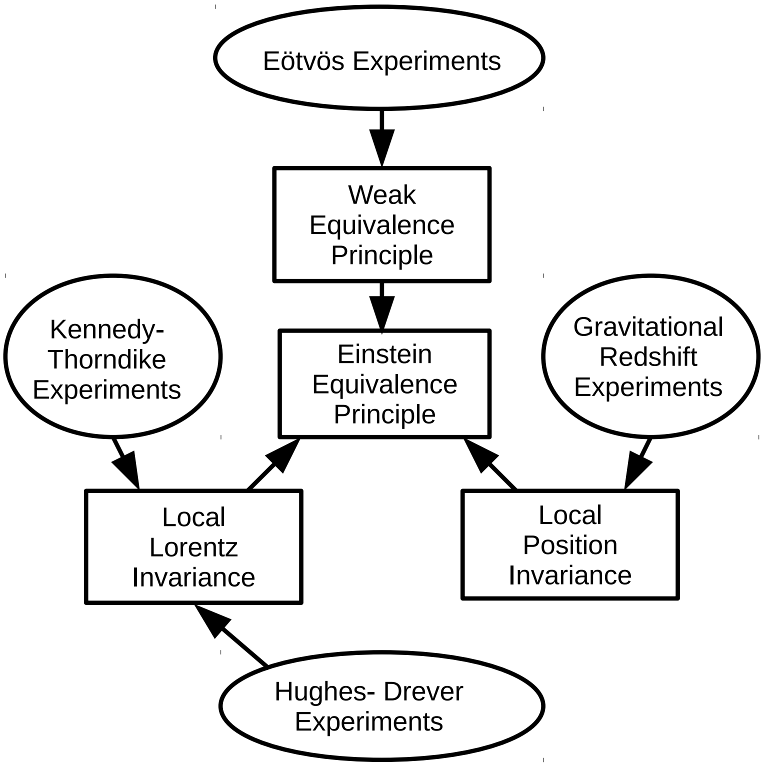

Figure 3: Schematic representation of the different experiments performed to test the three pillars of the EEP. As can be seen, every type of experiment tests only one single assumption in this simple picture, but it will be shown afterwards that if Schiff's conjecture is correct, some of the experiments test several points of the EEP, e.g. the Eötvös experiment is a test of all the three pillars at the same time.

Substituting the expression for the acceleration the test body X, which allows a violation of the WEP, given in equation (7) in to the definition of the Eötvös ratio yields

$$\eta(X,Y) = \frac{2\sum_A \eta^A \left(\zeta_X^A - \zeta_Y^A\right)}{2 + \sum_A \eta^A \left(\zeta_X^A + \zeta_Y^A\right)}. \tag{10}$$

As we know by experimental confirmation that the violation of WEP is very small $\eta^A \ll 1$, the second term in the denominator can be neglected and we may make the approximation

$$\eta(X,Y) \simeq \sum_A \eta^A \left(\zeta_X^A - \zeta_Y^A\right). \tag{11}$$

If we assume that the violations of the WEP by the different forms of energy do not compensate each other or that only one single form of energy A couples non metrically to gravity and thus violates the WEP, we can deduce the following relation between the Eötvös ratio and the WEP-violation parameter η^A

$$|\eta(X,Y)| \geq \left|\eta^A \left(\zeta_X^A - \zeta_Y^A\right)\right|, \tag{12}$$

from which an expression for the upper limit on the strength of violation of the WEP by energy of form A

$$|\eta^A| \leq \frac{|\eta(X,Y)|}{\left|\zeta_X^A - \zeta_Y^A\right|}, \tag{13}$$

can be deduced. Limits on the Eötvös ratio directly lead to limits on the strength of violation of the WEP. It is important to notice that the value of η depends on the experimental settings used (e.g. the internal structure and composition of the test bodies or the object used as an attractor). These limits are therefore not universal but depend mainly on the test bodies used. The limits placed on the WEP-violation parameter η^A however, are universal and can be directly compared to each other. If the WEP is valid, $\eta^A = 0$ which leads to $\eta = 0$, finding lower bounds on η and especially on the different η^A's supports the confirmation of the WEP.

A restriction on the violation of the WEP by $|\eta| \leq 10^{-3}$ for various substances was already found in the 17th century. Galileo Galilei tested the UFF by using masses of different compositions bound to wires of the same length, and comparing how long the pendulums kept step with each other. And Sir Isaac Newton by performing different pendulum experiments which are reported in his *Principia* [28]. Such pendulum experiments were later improved to an accuracy of a few parts in 10^5 by e.g Friedrich W. Bessel in 1832 [4] or Harold H. Potter in 1932 [35]. These limits could be drastically improved after the invention of the torsion balance, which can be described as a rod, on which the two test bodies are fixed, on a wire. If the two test bodies experience a different acceleration, there will be a torque on the wire which will be modulated if the entire apparatus is rotated. Roland von Eötvös, Desiderius Pekár and Eugen Fekete could fix $|\eta| \leq 5 \times 10^{-9}$ at the beginning of the 20th century, by using the Earth as an attractor and rotating the apparatus about the direction of the wire [48]. Highest limits up to the 1970's were set by the Princeton experiment performed by P. G. Roll, R. Krotov and Robert H. Dicke which measured $|\eta(\mathrm{Al},\mathrm{Au})| \leq 2.3 \times 10^{-11}$ [39], and the Moscow experiment[10] by Vladimir B. Bragnskiĭ and Vladimir I. Panov yielding $|\eta(\mathrm{Al},\mathrm{Pt})| \leq 1.3 \times 10^{-12}$ [6]. These experiments used the Sun as an attractor and the rotation of the Earth as the modulation. After these experiments, the use of rotating torsion balances could further improve the measured limits. The best limits obtained by experiments which were performed on Earth are set by the so called Eöt-Wash experiments performed by the University of Washington, whose most stringent limits are given by $|\eta(\mathrm{Be},\mathrm{Ti})| \leq 2.1 \times 10^{-13}$ and $|\eta(\mathrm{Be},\mathrm{Al})| \leq 2 \times 10^{-13}$ [49]. The MICROSCOPE mission,

[10] Both of these limits and all the following are given in 1σ-statistical uncertainty.

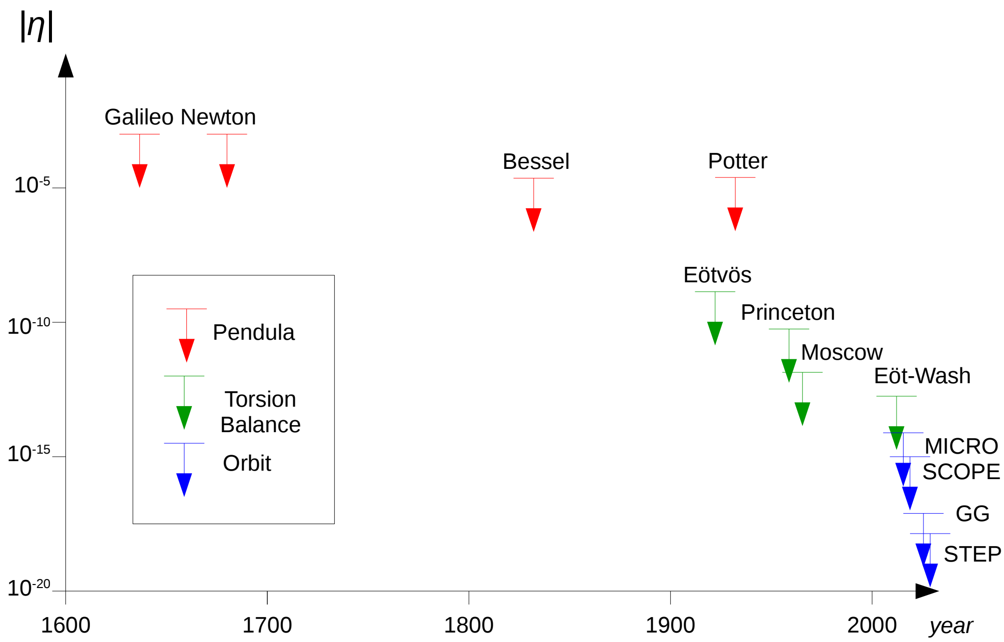

Figure 4: Illustration of the limits placed on the Eötvös ratio from the experiments listed in Table 1. The last three limits, namely the second limit of the MICROSCOPE mission and the limits from the GG and STEP mission are planned but not obtained yet. The improvements of the limits arising from new experimental techniques is clearly visible, as pendulum and torsion balance experiments do not reach values below 10^{-5} and 10^{-13}, respectively.

which was launched on the 25th of April 2016 and tested the WEP in the orbit by injecting a a satellite in a sun-synchronous circular orbit aan altitude of 710km, yielded a first value for $\eta(\mathrm{Ti, Pt}) = (-1 \pm 9(\mathrm{stat}) \pm 9(\mathrm{syst})) \times 10^{-15}$ in December 2017, which sets the upper limit to $|\eta(\mathrm{Ti, Pt})| \leq 1.3 \times 10^{-14}$ [44]. This is the lowest value up to date, but the mission aims to improve the upper bound to one part in 10^{15}. The planned small satellite Galileo Galilei (GG) mission aims to get to $|\eta| \leq 10^{-17}$ by orbiting at an altitude of 600 km around the Earth [31] and the planned Satellite test of the Equivalence Principle (STEP) mission aims to get to 1 part in 10^{18} [34]. A recent and detailed description of the historical progress in testing the WEP and an outlook on future progress is summarized in an article by Anna M. Nobili and Alberto Anselmi [30]. Some of the most important results obtained or hoped to be reached in future for the upper limit on $|\eta|$, are summarized in Table 1 and illustrated in Figure 4.

There have also been experiments to test the WEP with test bodies that have a non-negligible amount of self-gravitational energy, therefore testing the GWEP. The lunar laser ranging yielded $|\eta| \leq 2.1 \times 10^{-13}$ for any possible inequality in the ratios of the gravitational and inertial masses

Table 1: Summary of some of the historically most important and recent Eötvös experiments described in the text and the upper limits on the value of the Eötvös ratio $|\eta|$ they set. For the experiments of which the substances used as test bodies are clearly defined (Princeton, Moscow, Eöt-Wash and MICROSCOPE), limits on various violation parameters can be deduced. These are the experiments on which we will focus ourselves in the further analysis of this thesis.

Experiment	**Year**	**Reference**	**Method**	**Substances**	**Attractor**	**Limits on $|\eta|$**
Galileo Galilei	1638	[17]	Pendula	various	Earth	$\leq 10^{-3}$
Sir Isaac Newton	1686	[28]	Pendula	various	Earth	$\leq 10^{-3}$
Friedrich W. Bessel	1832	[4]	Pendula	various	Earth	$\leq 2 \times 10^{-5}$
Eötvös et al.	1922	[48]	Torsion Balance	various	Earth	$\leq 5 \times 10^{-9}$
Harold H. Potter	1932	[35]	Pendula	various	Earth	$\leq 2 \times 10^{-5}$
Princeton	1964	[39]	Torsion Balance	Al/Au	Sun	$\leq 2.3 \times 10^{-11}$
Moscow	1972	[6]	Torsion Balance	Al/Pt	Sun	$\leq 1.2 \times 10^{-12}$
Eöt-Wash (Be/Ti)	2012	[49]	Rotating Torsion Balance	Be/Ti	Sun	$\leq 2.1 \times 10^{-13}$
Eöt-Wash (Be/Al)	2012	[49]	Rotating Torsion Balance	Be/Al	Sun	$\leq 2 \times 10^{-13}$
MICROSCOPE	2017	[44]	Free Fall in Orbit	Ti/Pt	Earth	$\leq 1.3 \times 10^{-14}$ [1)]
GG[2)]	planned	[31]	Rotating Torsion Balance in Orbit	to be decided	Earth	$\leq 10^{-17}$
STEP[2)]	planned	[34]	Free Fall in Orbit	Pt/Ir/Nb/Be	Earth	$\leq 10^{-18}$

[1)] The aims of the MICROSCOPE missions is to get to $\eta \leq 10^{-15}$. [2)] These experiments are planned but have not been performed yet.

for the Earth and Moon [52]

$$\eta = \left(\frac{m_G}{m_I}\right)_{\mathrm{E}} - \left(\frac{m_G}{m_I}\right)_{\mathrm{M}} = (-0.8 \pm 1.3) \times 10^{-13}. \tag{14}$$

This type of experiments try to confirm the SEP instead of the EEP. The "Antimatter Experiment: Gravity, Interferometry, Spectroscopy" (AEgIS) experiment tries to confirm the WEP with antimatter [7].

3.2 Tests of Local Lorentz Invariance

If there is a violation of LLI of any form of energy A, one could expect a contribution to the inertial mass δm_I^{ij}, called anomalous inertial mass tensor. It is of the form

$$\delta m_I^{ij} \sim \sum_A \delta^A \frac{E^A}{c^2}, \tag{15}$$

where δ^A is a dimensionless parameter for the strength of anisotropy induced by interaction A. A violation of LLI leads to preferred frame effects, and therefore a preferred frame exists which could possibly be the Cosmic Microwave Background (CMB). The bare violation of LLI can be written as

$$\delta_0^A = \left(\frac{c}{w}\right)^2 \delta^A, \tag{16}$$

where w is the velocity of the laboratory with respect to the preferred frame. In the case of the CMB $w/c = 1.23 \times 10^{-3}$ or $w = 369$ km/s (values obtained from the Planck mission [1]). Limits on the strength of LLI violation can therefore be inferred from limits on δm_I^{ij} by assuming, as in the case of the WEP, that only one form of energy violates LLI, or at least that the different violations do not compensate each other

$$|\delta^A| \leq \frac{|\delta m_I^{ij} c^2|}{|E^A|}. \tag{17}$$

By dividing equation (15) by the mass of the test body X, we arrive at an expression including the fractional energy contribution ζ_X^A instead of the energy E^A

$$\frac{\delta m_I^{ij}}{m_X} = \sum_A \delta^A \zeta_X^A. \tag{18}$$

This will be useful to relate violations of the WEP and LLI.

For the case of the electromagnetic interaction, the c^2 formalism, which is a simplification of the $TH\epsilon\mu$ framework of Alan P. Lightman and David L. Lee [26] (see section 8), has been developed which allows a deviation from the speed of light $c \neq c_0$,where c_0 is its universal value. This formalism yields the following expression for the anomalous inertial mass tensor [51]

$$\delta m_I^{ij} c^2 = -2\delta \left[\frac{4}{3} E^{\mathrm{ES}} \delta^{ij} + \left(E^{\mathrm{ES}}\right)^{ij}\right], \tag{19}$$

where $E^{\mathrm{ES}}\delta^{ij}$ contributes to the isotropic and $\left(E^{\mathrm{ES}}\right)^{ij}$ to the anisotropic part of the anomalous inertial mass tensor. The dimensionless parameter δ is defined as

$$\delta \equiv \left(\frac{c_0}{c}\right)^2 - 1, \tag{20}$$

which is equal to zero if LLI is valid.

To include all the interactions of the standard model, Don Colladay and Alan Kostelecký developed a framework called the Standard Model Extension (SME) [10, 11]. The SME inserts a variety of tensorial quantities to the terms in the action of the standard SU(3) × SU(2) × U(1) field theory of particle physics. Experimental bounds on all the various parameters of the SME can be found in the review by Alan Kostelecký and Neil Russell [24]. This framework will not be discussed any further here since it is beyond the scope of this thesis.

In 1964 Vernon W. Hughes and his collaborators and Ronald Drever both measured the following limit on a possible anisotropy of the inertial mass $|\delta m^{ij}_{I,\text{aniso}}c^2| \leq 1.7 \times 10^{-16}$ eV from nuclear magnetic resonance experiments [22, 13]. The experiments included a ^{7}Li nucleus of which the energy splitting between the different J-levels were compared. If any preferred frame effects were present, the splitting would not have been constant. After 1985 J.D. Prestage [38] S. K. Lamoreaux [25] and T. E. Chupp and their collaborators [9] tested LLI and could improve the upper limits by performing atomic physics experiments using laser-cooled trapped atom techniques. The best limits reached were $|\delta m^{ij}_{I,\text{aniso}}c^2| \leq 2.1 \times 10^{-21}$eV, using ^{201}Hg. These experiments have been improved up to now, e.g. F. Allmendiger and his collaborators could fix the upper bound to $|\delta m^{ij}_{I,\text{aniso}}c^2| \leq 6.7 \times 10^{-25}$eV by comparing the energy splitting in ^{3}He and ^{129}Xe [2]. All these experiments set very stringent limits on the anisotropic parts of δm^{ij}. But to test the scalar part of δm^{ij} another experiment is needed. This is for example the Kennedy-Thorndike experiment, which is a modified form of the classical Michelson-Morley experiment, in which one arm is much shorter. In 1932 Roy J. Kennedy and Edward M. Thorndike could set an upper limit on the isotropic part of δm^{ij}_I[11] [23]

$$\frac{\delta m_{I,\text{iso}}c^2}{E^{\text{B}}} \leq 10^{-2}, \tag{21}$$

where E^{B} is the binding energy of the test body. The Turner-Hill experiment, named after their inventors K. C. Turner and H. A. Hill, could lower the limit to 10^{-4} [46]. More recent experiments with an increased precision, which was obtained by using lasers, masers, and cryogenic optical resonators, called cavity tests could set more stringent limits. For example M. E. Tobar and his collaborators reached an upper bound of 5.7×10^{-8} [43]. Thus one can see that the limits on a possible anisotropy of the inertial mass are of the order of 10^{25} times more stringent than any scalar deviations from it.

3.3 Tests of Local Position Invariance

The principle of LPI can be tested by the gravitational redshift experiment using clocks. Every type of clock depends either on oscillation frequencies between energy states or on the decay rate of a compound. This means that every type of clock depends on the transition of a certain form of energy and can therefore be used to test for a possible violation of LPI by this form of energy. The most common experiment to test LPI is to set two clocks in different gravitational potentials and comparing their ticking rates, to the one predicted by gravitational redshift of GR. This is referred to as the standard gravitational redshift experiment. This shift in frequency, referred to as gravitational redshift z is given by

$$z = \frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda}. \tag{22}$$

[11] In modern literature they use the parameter P_{KT} for these limits.

This frequency shift is a consequence of the first-order Doppler shift, and is given by

$$z = \frac{\Delta U}{c^2}, \tag{23}$$

where ΔU is the difference in Newtonian gravitational potential between the two positions of the clocks. If we allow a deviation from the gravitational redshift, it can be written as

$$z = (1+\alpha)\frac{\Delta U}{c^2}, \tag{24}$$

where α is a dimensionless parameter which measures the strength of the violation of LPI. Future gravitational redshift experiments (e.g the Atomic Clock Ensemble in Space (ACES) mission) may be expected to be able to detect the deviation from Newtonian gravitational potential, and one can test to a higher order in v^2/c^2. The gravitational potential can then be approximated by

$$U(r) = -1 + U_0(r) + U_1(r) = -1 - \frac{2Gm}{r} + \frac{2}{c^2}\left(\frac{Gm}{r}\right)^2, \tag{25}$$

which, by inserting this approximation in the expression given in equation (24), could give rise to new LPI violation parameters α_0 and α_1 that then can be tested separately. However, the last term U_1 is many orders of magnitude smaller than U_0, e.g for the experimental settings of the ACES mission (see below) one can approximate $|\Delta U_1/\Delta U_0| \simeq 10^{-9}$ or $|\Delta U_1/c^2| \simeq 10^{-19}$. The accuracy of the clocks is probably still not high enough to detect this term, since they are currently at a level of parts in 10^{-16}.

In 1960 R. V. Pound and G. A. Rebka measured the gravitational redshift with γ-photons of ^{57}Fe at the Jefferson Physical Laboratory tower at Harvard University up to an uncertainty of around 0.1 [36], which was improved until 1965 by R. V Pound and J. L. Snider to an uncertainty of around 0.01 yielding $|\alpha| \leq 10^{-2}$ [37]. These high accuracies were reached by making use of the Mössbauer effect. The most precise standard gravitational redshift experiment up to now was performed in 1979 and is called the Gravity Probe A (GPA) experiment or Vessot-Levine rocket experiment. It was based on the direct comparison between two hydrogen-masers, one placed on a spacecraft launched nearly vertically upwards to a height of around 20'000 km and the other one on the ground. Their result was an agreement between experiment and theory at a level of 7×10^{-5}, which led to the limit $|\alpha| \leq 2 \times 10^{-4}$ after the analysis of the data [47]. The ACES mission aims to measure the gravitational redshift up to a relative uncertainty of 2×10^{-6} which would set the upper limit on $|\alpha|$ on a few parts in 10^{-6} [40]. They will operate a new generations of clocks reaching a stability of a few parts in 10^{-16} on the International Space Station, which orbits at an altitude of 400 km. It contains a cesium frequency standard based on laser-cooled atoms called Project d'Horologe Atomique par Refroidissement d'Atomes en Orbit (PHARAO), developed by the Centre National d'Études Spatiales (CNES), France, and a space hydrogen maser, developed by the Neuchâtel Observatory and Spectratime in Switzerland.

A different type of experiment used to test LPI are the so called "null"-redshift experiments. These test different type of clocks at the same position and look if their relative rates depend upon the gravitational potential. For example the comparison of two hydrogen maser clocks and three Superconducting-Cavity Stabilized Oscillator (SCSO) clocks yielded the limit [45]

$$|\alpha_{\rm H} - \alpha_{\rm SCSO}| \leq 1.7 \times 10^{-2}, \tag{26}$$

whereas the comparison between hyperfine transitions in ^{87}Rb and ^{133}Cs yielded [18]

$$|\alpha_{\rm Rb} - \alpha_{\rm Cs}| \leq 1.2 \times 10^{-6}. \tag{27}$$

If LPI is valid, the laws of physics are not only independent of the position, but of time, too. This leads to the constraint that non-gravitational constants must be constant in time.

4 Schiff's Conjecture

In 1960, Leonard I. Schiff conjectured that the three tests of the EEP are not independent of each other [41]. The confirmation of the WEP is by the same time a confirmation of LLI and LPI, one might say that the WEP implies the EEP and the metric postulate. Or stated differently, if WEP is valid, the theory of gravitation must be a metric one. Different proofs of Schiff's conjecture were tried (see e.g. the restricted proof of Alan P. Lightman and David L. Lee [26]) but it is in fact impossible to proof such a conjecture. Most of the arguments depend on the principle of energy conservation, which was already used in this context by Albert Einstein as an argument for the existence of the gravitational redshift [15]. This will also be the starting point for the two derivation given in this section. There have been found several counterexamples, e.g. by Wei-Tou Ni, too [29]. The validity of Schiff's conjecture is therefore still questioned. In Figure 5 a schematic representation of the three pillars of the EEP and their connection with each other are given. Also included are the most important experiments described in section 3, which test them.

The first part of this section will be based on a cyclic thought experiment similar to the one mentioned by Kenneth L. Nordtvedt Jr. in 1975 [33]. This will result in a quantitative relationship between violations of the WEP and LPI, or to be more concrete, between Eötvös and gravitational redshift experiments (see the red arrow on the left side in Figure 5). The second part will include violations of LLI, by performing a thought experiment based on the one done by Mark P. Haugan in 1979 [19] (see the red arrow on the right side in Figure 5).

4.1 Quantitative Relationship between Violations of the Weak Equivalence Principle and Local Position Invariance

In this subsection, a simple cyclic thought experiment based on energy conservation will be presented, which follows the work of Kenneth L. Nordtvedt Jr. done in 1975, which he used to calculate a quantitative relationship between the violation of the WEP and corresponding violations of gravitational redshift experiments [33].

Cyclic thought experiment: Two different test bodies X and Y, with corresponding masses m_X and m_Y respectively, are placed in a uniform gravitational field aligned in the z-direction. Both of them are at rest ($v = 0$), the test body X is placed at an altitude $z = 0$ and Y at a height h (see Figure 6 a)). Imagine now that the test body Y makes a transition to X by emitting a quantum q containing an energy E_q given by

$$E_q = \Delta mc^2 = (m_Y - m_X)c^2, \tag{28}$$

where the nature of the emitted quantum is not specified any further, it could for example be a photon, a gluon or an α-particle (see Figure 6 b))[12]. This quantum q now travels downwards in the gravitational field from $z = h$ to $z = 0$ and is absorbed there by the test body X. This absorption leads to a transition from X to Z, so that we end up with two test bodies X and Z at $z = h$ and $z = 0$ respectively, where Z has a corresponding mass m_Z (see Figure 6 c)). The test body X at height h now free falls with an acceleration a_X in the negative z-direction. It reaches a velocity v_X at $z = 0$ which can be expressed by

$$v_X^2 = 2a_X h, \tag{29}$$

[12]Kenneth L. Nordtvedt Jr. used two identical quantum systems in different states, instead of two different test bodies. But as the nature of the quantum which is emitted in the transition from Y to X is not restricted, this process can be extended to any two test bodies.

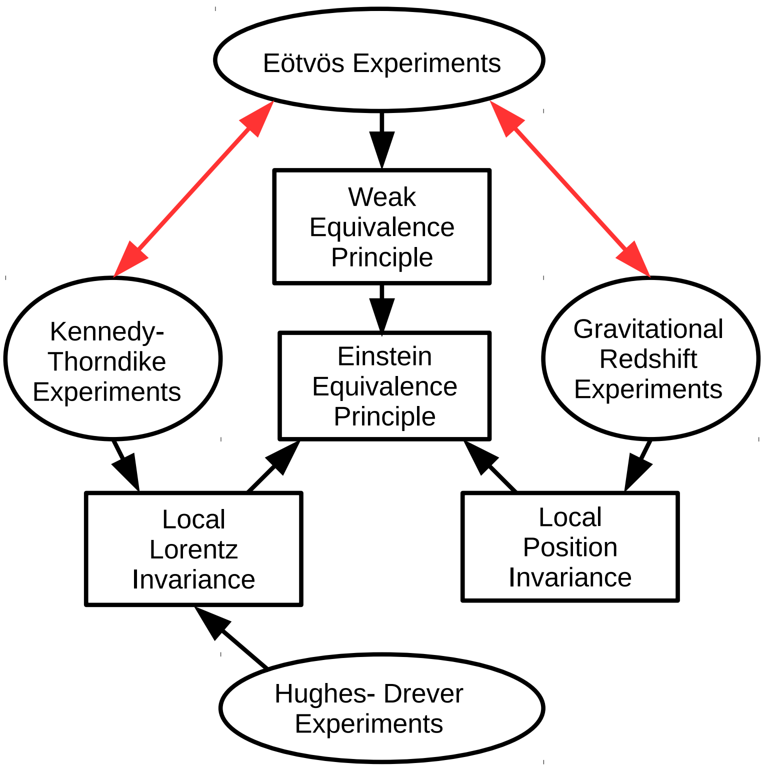

Figure 5: Schematic representation of the three pillars of the EEP, their dependencies on experiments as well as on each other. The three pillars of the EEP are tested via different type of experiments, but as we will see in this section, there exists a quantitative relationship between the Eötvös experiments and Kennedy-Thorndike or gravitational redshift experiments (see red arrows). If Schiff's conjecture is valid, the Eötvös experiment can be seen as a test of all the three assumptions of the EEP at the same time, and therefore a confirmation of the WEP would directly imply the EEP.

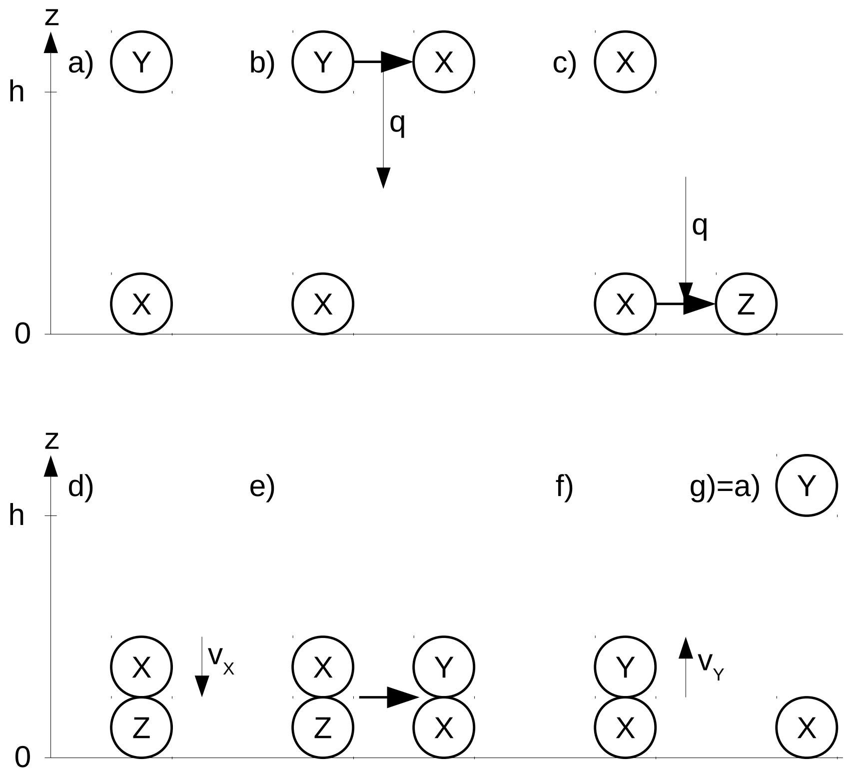

Figure 6: Illustration of the cyclic thought experiment based on the work of Kenneth L. Nordtvedt Jr. described in the text [33]. It is used to find a relation between violations of the WEP and LPI.

(see Figure 6 d)). There it inelastically collides with the test body Z, leaving again two test bodies X and Y (see Figure 6 e)). All the leftover energy of this collision is given to the test body Y, which then travels upwards in the gravitational potential and experiences a deceleration a_Y[13]. Due to the assumption of energy conservation it must be able to reach a height $z = h$ and therefore needs a velocity v_Y at $z = 0$ given by

$$v_Y^2 = 2a_Y h, \tag{30}$$

(see Figure 6 f)). Thus one full cycle is fulfilled and we are again at the starting point of the cyclic thought experiment (see Figure 6 g)).

If we take a closer look at the different energy contributions present at the point of collision we can write

$$\frac{1}{2}m_X v_X^2 + m_X c^2 + m_Z c^2 = m_X c^2 + \frac{1}{2}m_Y v_Y^2 + m_Y c^2, \tag{31}$$

[13]We allow a deviation from the WEP, therefore the two test bodies X and Y may experience a different acceleration a due to their internal structure or composition.

where on the left and on the right side the kinetic and rest mass energies before and after the inelastic collision are summed up, respectively. Due to the principle of conservation of energy, these two sides must be equal to each other. This can be rewritten using the expressions for the velocities v_X and v_Y given in equations (29) and (30), and by canceling out the rest mass energy of X on both sides, as

$$m_X a_X h + m_Z c^2 = m_Y a_Y h + m_Y c^2. \tag{32}$$

Our aim is to find a quantitative relation between the violation of the gravitational redshift and the violation of the UFF. First of all we rewrite the equation above as an expression for the mass difference between the test bodies Z and Y which arises due to the gravitational redshift of the quantum q

$$m_Z - m_Y = \frac{h}{c^2}(m_Y a_Y - m_X a_X) = \frac{h}{2c^2}\left[(m_Y - m_X)(a_Y + a_X) + (m_Y + m_X)(a_Y - a_X)\right]. \tag{33}$$

The gravitational redshift is defined as the shift in frequency divided by the frequency itself (see equation (22)) or stated differently, as the difference between the received and the emitted energy divided by the emitted energy. In our cyclic thought experiment the emitted energy corresponds to the energy of the quantum q at an altitude $z = h$, whereas the received energy is its energy at $z = 0$

$$z = \frac{E_{\text{received}} - E_{\text{emitted}}}{E_{\text{emitted}}} = \frac{(m_Z - m_X)c^2 - (m_Y - m_X)c^2}{(m_Y - m_X)c^2} = \frac{m_Z - m_Y}{m_Y - m_X}. \tag{34}$$

The last term can be easily related to the expression given in equation (33) and we obtain the following connection between the gravitational redshift and the thought experiment

$$z = \frac{h}{2c^2}\left[a_Y + a_X + (a_Y - a_X)\frac{m_Y + m_X}{m_Y - m_X}\right]. \tag{35}$$

If we introduce the gravitational acceleration g as the average between the free fall accelerations of the test bodies X and Y, meaning $g = (a_Y + a_X)/2$ we can write

$$z = \frac{gh}{c^2}\left[1 + \frac{a_Y - a_X}{a_Y + a_X}\frac{m_Y + m_X}{m_Y - m_X}\right]. \tag{36}$$

By identifying the ratio between the accelerations with the Eötvös ratio as defined in equation (9) and allowing deviations from LPI by writing the gravitational redshift as given in equation (24) we arrive at

$$(1+\alpha)\frac{\Delta U}{c^2} = \frac{\Delta U}{c^2}\left[1 + \eta(Y,X)\frac{m_Y + m_X}{2(m_Y - m_X)}\right], \tag{37}$$

where the difference in gravitational potential $\Delta U = gh$ is used. From this equation, a simple quantitative relationship between the LPI violation parameter α and the Eötvös ratio η can be deduced

$$\alpha\frac{2(m_X - m_Y)}{m_X + m_Y} = \eta(X,Y). \tag{38}$$

If we assume that the two test bodies have about the same mass $m_X \sim m_Y$ we can write the result in the following final form

$$\boxed{\eta(X,Y) = \alpha^A(\zeta_X^A - \zeta_Y^A),} \tag{39}$$

where A denotes the form of energy which is assumed to violate LPI and is transferred in the gravitational redshift experiment from which the limit on the parameter α is extracted (e.g.

hyperfine energy if a hydrogen maser is used as clock). The fractional energy contribution ζ was already defined in equation (8).

Using the relation between the Eötvös ratio and the parameter for the strength of the violation of the WEP of interaction A given in equation (13), we can find an expression for the limit set on the WEP violation parameter by the gravitational redshift experiments

$$|\eta^A| \leq \left| \frac{\zeta_X^B - \zeta_Y^B}{\zeta_X^A - \zeta_Y^A} \right| |\alpha^B|, \tag{40}$$

which for the simplest case, where the two forms of interactions are identical $A = B$ simplifies to

$$|\eta^A| \leq |\alpha^A|. \tag{41}$$

4.2 Quantitative Relationship between Violations of the Weak Equivalence Principle and Local Lorentz Invariance

This subsection follows the work done by Mark P. Haugan in 1979, in which he related violations of LLI and LPI to a corresponding violation of the WEP on the basis of the principle of energy conservation [19]. He thus ended up with a quantitative expression for the relation between the three pillars of the EEP. Here we will only consider the connection between the violation of LLI and the WEP, since the latter was already derived in the previous subsection.

The total energy E_{tot} of a non-gravitationally bound test body X in the uniform gravitational potential $U(z)$ can be summed up as

$$E_{\text{tot},X} = m_{0,X}c^2 - E_X^B(v) - m_X U(z) + \frac{1}{2} m_X v_X^2, \tag{42}$$

where $m_{0,X}$ is the sum of the constituent masses, E_X^B is the binding energy of the test body and the last two terms are the potential and kinetic energy term, respectively. The height z and the velocity v are given in quasi-Newtonian coordinates of the center of mass. The velocity dependence in $E_X^B(v)$ allows a possible violation of LLI. It can be expanded in the center-of-mass velocity up to first order, thus yielding

$$E^B(v) = E_0^B - \frac{1}{2} \delta m_{I,\text{iso}}^{ij} v^2, \tag{43}$$

where $\delta m_{I,\text{iso}}^{ij}$ is the scalar part of the anomalous inertial mass tensor. The anisotropic part does not yield any contribution to the binding energy. The total energy, allowing a violation of LLI, can therefore be written as

$$E_{\text{tot},X} = m_{0,X}c^2 - E_{0,X}^B - m_X U(z) + \frac{1}{2} m_X v_X^2 + \frac{1}{2} \delta m_{I,\text{iso}}^{ij} v_X^2. \tag{44}$$

Thought experiment: Let a test body X initially be at rest $v_X = 0$ at a height $z = h$ in a uniform gravitational potential (see Figure 7 a)), then it possesses a total amount of energy given by

$$E_{\text{tot},X} = m_{0,X}c^2 - E_{0,X}^B - m_X U(h). \tag{45}$$

It then free falls to $z = 0$ reaching a final velocity $v_{f,X}$ at $z = 0$ (see Figure 7 b)). The total energy at this point can then be summed up as

$$E_{\text{tot},X} = m_{0,X}c^2 - E_0^B - m_X U(0) + \frac{1}{2} m_X v_{f,X}^2 + \frac{1}{2} \delta m_{I,\text{iso}}^{ij} v_{f,X}^2. \tag{46}$$

Due to the principle of energy conservation, the total energy must be constant and therefore we can write

$$\frac{1}{2}m_X v_{f,X}^2 = -m_X \left[U(h) - U(0)\right] - \frac{1}{2}\delta m_{I,\mathrm{iso}}^{ij} v_{f,X}^2. \tag{47}$$

As we suppose that we are in uniform gravitational potential, the acceleration a is constant and the velocity at $z = 0$ is simply given by $v_f^2 = -2ah$. By expanding $U(z)$ in first order in z around $z = 0$

$$U(z) = U(0) + \nabla U|_{z=0} z \tag{48}$$

leads, by using $\nabla U = g$, to

$$-m_X a_X h = -m_X g h + \delta m_{I,\mathrm{iso}}^{ij} a h. \tag{49}$$

For the isotropic part of the inertial mass tensor, one finds the following relation to the acceleration of a test body X

$$a_X = \left(1 - \frac{\delta m_{I,\mathrm{iso}}^{ij}}{m_X}\right) g = \left(1 - \sum_A \delta_{\mathrm{iso}}^A \zeta_X^A\right) g, \tag{50}$$

where equation (18) was used in the second step. Inserting this expression into the definition of the Eötvös parameter in equation (9), we find the following relationship to the violation of LLI

$$\boxed{\eta(X,Y) = -\delta_{\mathrm{iso}}^A \left(\zeta_X^A - \zeta_Y^A\right),} \tag{51}$$

under the assumption that only a single form of energy violates LLI in the experiment. We can deduce the following relation

$$|\eta^A| \leq \left|\frac{\zeta_X^B - \zeta_Y^B}{\zeta_X^A - \zeta_Y^A}\right| |\delta_{\mathrm{iso}}^B|, \tag{52}$$

which for the simplest case, where the two types of interactions are identical $A = B$ simplifies to

$$|\eta^A| \leq |\delta_{\mathrm{iso}}^A|. \tag{53}$$

Notice the strong analogy of equations (51) to (53) describing the relation between violations of LLI and the WEP, with equations (39) to (41) doing the same for LPI.

If we allow both LLI and LPI violations we end up with the following relationships between all the three pillars of the EEP by simply combining the expressions given above

$$\boxed{\eta(X,Y) = \alpha^A(\zeta_X^A - \zeta_Y^A) - \delta_{\mathrm{iso}}^B(\zeta_X^B - \zeta_Y^B).} \tag{54}$$

If we assume that the different pillars of the EEP are violated by the same form of energy A we end up with

$$|\eta^A| \leq |\alpha^A - \delta_{\mathrm{iso}}^A|, \tag{55}$$

which is a plausible assumption, since if there is a non metric coupling of a single form of energy, it is probable that it violates more than one pillar of the EEP. The assumption that one form of energy violates only a single pillar, whereas another form violates another one, seems to be less promising.

The next sections will be concerned with how to use these equations to calculate limits on the different parameters. However, this calculations always underly some assumptions on the coupling between the different interactions to gravity one must do and are therefore to be handled with caution. The first thing we will have a look at in the following section is a way to calculate the energies and energy contributions of different forms.

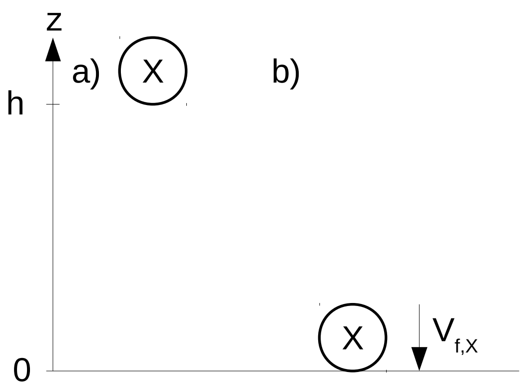

Figure 7: Illustration of the thought experiment mentioned in the text to get a quantitative relation between violations of the WEP and LLI. It is based on the work done by Mark P. Haugan [19].

5 Different Forms of Energy

If we take a look at equations (13) and (17) we see that, to calculate limits on the WEP and LLI-violation parameters for different forms of energy, we must find an expression for the energy E^A as well as the fractional energy contributions ζ_X^A. The substances which were used in recent or historically important Eötvös and Hughes-Drever experiments are listed in Table 2. For the Kennedy-Thorndike and the gravitational redshift experiments, the substances are not of such an importance for us because the violation parameters can be directly deduced from the results which were measured.

For laboratory-sized bodies, the contribution to E^A is dominated by the atomic nucleus, since the mass of the electrons in an atom is much small than the mass of the nucleons $m_e/m_{\text{nucleon}} \sim 1/1836$. We will use the semi-empirical mass formula (sometimes referred to as Weizsäcker's formula), which describes the binding energy of an atomic nucleus through the following terms

$$E^{\text{B}} = m_0 - m, \tag{56}$$

where m denotes the rest mass of a test body and m_0 the sum of its constituent masses. This formula yields a sum of several contributions to the nuclear binding energy[14] [12]

$$E^{\text{B}} = E^{\text{V}} + E^{\text{Sf}} + E^{\text{As}} + E^{\text{ES}} + E^{\text{P}}. \tag{57}$$

The first term of this sum is called the volume energy term E^{V} and is given by

$$E^{\text{V}} = -16.0A \text{ MeV}, \tag{58}$$

[14] There are other contributions to the binding energy which are not listed in equation (57) because they are a few orders of magnitude smaller. We will especially have a look at the magnetostatic, the hyperfine and the weak interaction which are not listed here.

Table 2: List of the substances and isotopes used as test bodies in the most recent or accurate Eötvös and Hughes-Drever experiments. Their corresponding natural abundance, atomic number Z, number of neutrons N and mass number A are given, too. If for the test body only the element is specified and not the isotope, an average over all its stable isotopes and their corresponding natural abundances is taken. For the Eötvös the fractional energy contributions ζ^A and for the Hughes-Drever experiments the energies E^A need to be calculated, respectively.

Substance	**Isotope**[1)]	**Natural Abundance**	Z	N	A	**Experiment**[2)]
Helium	^{3}He	<0.01%	2	1	3	[2] (HD)
Lithium	^{7}Li	92.6%	3	4	7	[22] (HD)
Beryllium	^{9}Be	100%	4	5	9	[49] (E)
Aluminum	^{27}Al	100%	13	14	27	[39, 6, 49] (E)
Titanium	^{46}Ti	8%	22	24	46	[49, 44] (E)
	^{47}Ti	7.3%	22	25	47	
	^{48}Ti	73.8%	22	26	48	
	^{49}Ti	5.5%	22	27	49	
	^{50}Ti	5.4%	22	28	50	
Xenon	^{129}Xe	26.4%	54	75	129	[2] (HD)
Platinum	^{192}Pt	0.79%	78	114	192	[6, 44] (E)
	^{194}Pt	32.9%	78	116	194	
	^{195}Pt	33.8%	78	117	195	
	^{196}Pt	25.3%	78	118	196	
	^{198}Pt	7.2%	78	120	198	
Gold	^{197}Au	100%	79	118	197	[39] (E)
Mercury	^{199}Hg	16.87%	80	119	199	[25] (HD)
	^{201}Hg	13.18%	80	121	201	

[1)] For simplicity, it is assumed that only the stable isotopes contribute significantly to the test bodies in the experiments. [2)] The labels (E) or (HD) denote that the substance was used as a test body in an Eötvös or Hughes-Drever experiment, respectively.

where A is the atomic mass number[15]. This term arises from the strong interaction between the nucleons. Since the volume of a nucleus is proportional to its mass number, this term is proportional to the volume of the nucleus, which explains its name. It is independent of Z as both, protons and neutrons are affected by the strong interaction. The second term of the semi-empirical mass formula is the surface energy term E^{Sf} which can be understood as a correction to the volume energy term. It is given by

$$E^{\mathrm{Sf}} = 17.0 A^{2/3} \ \mathrm{MeV}, \tag{59}$$

which is proportional to the surface of a nucleus. It arises due to the fact that nucleons at the surface of a nucleus do interact with less neighboring nucleons via the strong interaction, than a nucleon at the center. The third term, which is called the asymmetry energy term E^{As} exists because of the Pauli exclusion principle, which states that two identical fermions can not occupy the same quantum state. Since protons and neutrons do not have the same quantum states, the binding energy which will be the highest, if their numbers are similar. The term therefore depends on the difference in the number of protons and neutrons and is given by

$$E^{\mathrm{As}} = 23.0 \frac{(N-Z)^2}{A} \ \mathrm{MeV}, \tag{60}$$

where Z is the atomic and N the neutron number, respectively. The repulsive Coulomb or electrostatic energy term E^{ES} is given by

$$E^{\mathrm{ES}} = 0.7 \frac{Z(Z-1)}{A^{1/3}} \ \mathrm{MeV}. \tag{61}$$

This term is simply the electrostatic repulsion between the protons. The last term of equation (57) is known as the pairing energy term E^{P} and is given by

$$E^{\mathrm{P}} = -12.0 \frac{\delta}{A^{1/2}} \ \mathrm{MeV}, \tag{62}$$

where $\delta = 1$ if N and Z are even numbers, $\delta = -1$ if they are odd, and $\delta = 0$ if one is even and the other one is odd. Due to the Pauli exclusion principle, the binding energy will be the highest, if the number of protons and neutrons with spin up, is equal to the ones with spin down. But as this is only possible if the number of protons and neutrons are even, odd numbers will lower the binding energy. Other forms of energy which will be analyzed are the nuclear magnetostatic E^{MS}, the hyperfine E^{HF} and the weak energy E^{W}. The contribution of the self-gravitational energy to the total energy will not be considered since for an atomic nucleus [50]

$$\zeta^{\mathrm{G}}_{\mathrm{nucleus}} \sim \frac{G m_{\mathrm{nucleus}}}{c^2 r_{\mathrm{nucleus}}} \sim 10^{-39}, \tag{63}$$

and is therefore negligibly small. One may expect that all the various energy terms above contribute separately to violations of the EEP and thus could have different η^A's and δ^A's [3].

To get the fractional energy contributions ζ^A out of the energy E^A, we may approximate the rest mass energy of an atom by $mc^2 \simeq 931.5A$ MeV and get the following relation

$$\zeta^A = \frac{E^A A^{-1}}{931.5 \ \mathrm{MeV}}. \tag{64}$$

[15] Not to be confused with the form of energy which is also denoted as A.

Table 3: Energy contributions of the strong interaction and their single terms for the substances which were used in the Hughes-Drever experiments mentioned in the text. The values are given in units of MeV.

Isotope	$E_X^{\mathbf{V}}$	$E_X^{\mathbf{Sf}}$	$E_X^{\mathbf{As}}$	$E_X^{\mathbf{S}}$
^{3}He	−48.0	35.4	7.67	−5.0
^{7}Li	−112	62.2	3.29	−47
^{129}Xe	−2060	435	78.6	−1550
^{199}Hg	−3180	580	176	−2430
^{201}Hg	−3220	583	192	−2440

5.1 The Strong Interaction

The terms of the semi-empirical mass formula given in equation (57) which arise from the strong interaction are the volume, the surface, the asymmetry and the pairing energy term. The total amount of strong energy can therefore be approximated by their sum

$$E^{\mathrm{S}} = E^{\mathrm{V}} + E^{\mathrm{Sf}} + E^{\mathrm{As}} + E^{\mathrm{P}} = -16.0A + 17.0A^{2/3} + 23.0\frac{(N-Z)^2}{A} - 12.0\frac{\delta}{A^{1/2}}\ \mathrm{MeV}. \quad (65)$$

If we take a look at the expression for the LLI violation parameter given in equation (17), we see that the amount of energy E^{A} must be known for the substances used in the Hughes-Drever experiment to calculate any limits. For example the experiment performed by Vernon W. Hughes and Ronald Drever included a ^{7}Li nucleus ($A = 7$, $Z = 3$, $N = 4$). Its amount of strong energy can be calculated and yields a value of $E^{\mathrm{S}}_{^7\mathrm{Li}} = -47$ MeV. For the other substances the energies of the strong interaction and its single contributions are listed in Table 3. We get the fractional energy contribution of the strong interaction to the total energy by the relation given in equation (64) on the expression of the amount of strong energy given above.

$$\begin{aligned}\zeta^{\mathrm{S}} &= \frac{1}{mc^2}\left(E^{\mathrm{V}} + E^{\mathrm{Sf}} + E^{\mathrm{As}} + E^{\mathrm{P}}\right) \\ &= \left[-1.72 + 1.83\frac{1}{A^{1/3}} + 2.47\left(1 - \frac{2Z}{A}\right)^2 + 1.29\frac{\delta}{A^{3/2}}\right]10^{-2}.\end{aligned} \quad (66)$$

The fractional energy contribution must be calculated for the substances used in various Eötvös experiments to get the limits on the WEP violation parameters as can be seen from equation (13). For example the difference in the fractional energy contribution of the strong interaction, if we average over all the stable isotopes with the corresponding natural abundance, between titanium and platinum is given by $\zeta^{\mathrm{S}}_{\mathrm{Ti}} - \zeta^{\mathrm{S}}_{\mathrm{Pt}} \simeq 4 \times 10^{-5}$, respectively. Values for all the substances can be found in Table 4.

5.2 The Electromagnetic Interaction

Here we will consider the following contributions arising from the electromagnetic interaction

$$E^{\mathrm{EM}} = E^{\mathrm{ES}} + E^{\mathrm{MS}} + E^{\mathrm{HF}}, \quad (67)$$

Table 4: Fractional energy contribution of the volume- ζ^{V}, the surface- ζ^{Sf}, the asymmetry- ζ^{As} and the pairing energy ζ^{P} as well as of the total strong interaction energy ζ^{S} for the different substances X used in the considered Eötvös experiments listed in Table 2. The fractional energy contribution of the volume energy to the total energy is constant for all the substances and the only substances which get a contribution from the pairing energy are titanium and platinum, because all the other isotopes do neither have both even or odd proton- and neutron numbers Z and N, respectively.

Substance	$\zeta_X^{\mathbf{V}} \times 10^2$	$\zeta_X^{\mathbf{Sf}} \times 10^3$	$\zeta_X^{\mathbf{As}} \times 10^4$	$\zeta_X^{\mathbf{P}} \times 10^3$	$\zeta_X^{\mathbf{S}} \times 10^2$
Beryllium	-1.72	8.77	3.05	0.00	-0.81
Aluminum	-1.72	6.08	0.338	0.00	-1.11
Titanium	-1.72	5.02	1.71	-1.62	-1.36
Platinum	-1.72	3.15	9.93	-0.610	-1.36
Gold	-1.72	3.14	9.68	0.00	-1.31

which are the nuclear electrostatic, magnetostatic and hyperfine energy, respectively. The total amount of energy in the electromagnetic form is simply approximated as the sum of these contributions. We will, however, see that by far the largest contribution arises from the electrostatic term and we can make the approximation $E^{\mathrm{EM}} \simeq E^{\mathrm{ES}}$. The amount of electrostatic nuclear energy and the fractional contribution of the electrostatic nuclear energy to the total energy of the test body can be calculated from the electrostatic energy term E^{ES} which is given in equation (61) and therefore

$$\zeta^{\mathrm{ES}} = 7.6 \times 10^{-4} Z(Z-1) A^{-4/3}, \tag{68}$$

respectively. The calculated values for the electrostatic nuclear energy contribution for gold and aluminum are $\zeta_{\mathrm{Au}}^{\mathrm{ES}} \simeq 1.5 \times 10^{-3}, \zeta_{\mathrm{Al}}^{\mathrm{ES}} \simeq 4.1 \times 10^{-3}$, which yields the following fractional difference $\zeta_{\mathrm{Au}}^{\mathrm{ES}} - \zeta_{\mathrm{Al}}^{\mathrm{ES}} \simeq 2.6 \times 10^{-3}$ between these two substances. If we compare this value to Kenneth L. Nordtvedt Jr., who calculated this difference in 1975 and obtained a value of $|\delta M/M| \simeq 4 \times 10^{-3}$ [33], we see that the deviation is only about half of its value and that the assumptions made to calculate the different amounts of energy are justified at least as rough approximations[16].

The magnetostatic nuclear energy arises from the nuclear magnetic fields generated by the proton currents. As the magnetostatic energy contribution depends on the shell structure of the nucleus, it is not that simple to calculate it. It was done for aluminum, gold and platinum by Mark P. Haugan and Clifford M. Will with the following results $\zeta_{\mathrm{Al}}^{\mathrm{MS}} = 4.1 \times 10^{-7}, \zeta_{\mathrm{Pt}}^{\mathrm{MS}} = 2.4 \times 10^{-7}, \zeta_{\mathrm{Au}}^{\mathrm{MS}} = 2.6 \times 10^{-7}$ [20]. They proposed, however, that the fractional energy contribution is approximately proportional to $A^{-1/3}$, that is why we will use the following expression

$$\zeta^{\mathrm{MS}} = 1.1 \times 10^{-6} A^{-1/3}, \tag{69}$$

which was obtained by calculations based on a linear regression. This leads to the following formula to approximate the magnetostatic nuclear energy

$$E^{\mathrm{MS}} \simeq 9.9 \times 10^{-4} A^{2/3} \text{ MeV}. \tag{70}$$

The energy arising from the interaction between the spins of the protons and neutrons and the magnetic fields generated by their magnetic moments, namely the hyperfine energy, can be

[16]Kenneth L. Nordtvedt used the mass difference $\delta M/M$ as a convention instead of the difference in fractional energy used in this thesis.

Table 5: The amount of energy of the electromagnetic interaction and its contributions for the isotopes used in the Hughes-Drever experiments of interest. The values are given in units of MeV. The magnetostatic energies were calculated by the linear model given in equation (70), following the work done by Mark P. Haugan and Clifford M. Will [20].

Isotope	$E_X^{\mathbf{ES}}$	$E_X^{\mathbf{MS}} \times 10^3$	$E_X^{\mathbf{HF}}$	$E_X^{\mathbf{EM}}$
^{3}He	0.97	2.1	0.28	1.20
^{7}Li	2.2	3.6	0.36	2.6
^{129}Xe	400	25	5.01	400
^{199}Hg	760	34	6.58	760
^{201}Hg	760	34	5.05	760

approximated by [50]

$$E^{\mathrm{HF}} = \frac{2\pi}{V}\mu_N^2 \left[\left(\frac{g_p}{2}\right)^2 Z^2 + \left(\frac{g_n}{2}\right)^2 N^2\right], \tag{71}$$

where V and μ_N are the nuclear volume and magneton, and $g_p = 5.59$ and $g_n = -3.83$ (values from [27]) are the g-factors for the proton and the neutron, respectively. The nuclear volume is proportional to the atomic mass number, evaluating the constants therefore leads to the following formula for the hyperfine energy

$$E^{\mathrm{HF}} = 0.153\frac{Z^2}{A} - 0.0719\frac{N^2}{A}\ \mathrm{MeV}. \tag{72}$$

The fractional energy contribution is derived by the same way as was done for all the other interactions, it is therefore given by

$$\zeta^{\mathrm{HF}} = \left[1.64\left(\frac{Z}{A}\right)^2 + 0.772\left(\frac{N}{A}\right)^2\right] 10^{-4}. \tag{73}$$

For example for titanium and platinum the values calculated by the expression are $\zeta^{\mathrm{HF}}_{\mathrm{Ti}} \simeq 5.71 \times 10^{-5}$ and $\zeta^{\mathrm{HF}}_{\mathrm{Pt}} \simeq 5.40 \times 10^{-5}$, respectively. With a corresponding difference in the range of $\zeta^{\mathrm{HF}}_{\mathrm{Ti}} - \zeta^{\mathrm{HF}}_{\mathrm{Pt}} \simeq 3.1 \times 10^{-6}$. Kenneth L. Nordtvedt Jr. made the assumption that all the magnetic interactions, including the hyperfine interaction, violate the EEP and he estimated a mass difference of $\delta M/M \simeq 5 \times 10^{-5}$, which is about an order of magnitude higher than what we expect. In an article from the Spacetime Explorer and Quantum Equivalence Principle Space Test (STE-QUEST) mission, the contribution of the hyperfine interaction to the total energy of an atom is assumed to by typically around $\zeta^{\mathrm{HF}} \simeq 10^{-16}$ [3], which is more than 10^{10} times less than what we expect here and would have a huge impact on the results if it were true. But in fact they only consider the amount of energy between different hyperfine levels and not the total hyperfine energy of an atomic nucleus. Values of the electromagnetic interaction for the substances used in the Hughes-Drever and Eötvös experiments are tabulated in Table 5 and 6, respectively.

5.3 The Weak Interaction

The weak force has a short range, just like the strong force. But as it is already obvious from its name, it is much weaker. Therefore its contribution to the total energy of an atomic nucleus is

Table 6: Fractional energy contribution of the different parts of the electromagnetic interaction. The magnetostatic contribution was calculated by Mark P. Haugan and Clifford M. Will [20]. The total fractional energy contribution of the electromagnetic interaction is simply the sum of its contributions.

Substance	$\zeta_X^{\mathbf{ES}} \times 10^3$	$\zeta_X^{\mathbf{MS}} \times 10^7$	$\zeta_X^{\mathbf{HF}} \times 10^5$	$\zeta_X^{\mathbf{EM}} \times 10^3$
Beryllium	0.48	5.2[1)]	5.62	0.5
Aluminum	1.5	4.1	5.87	1.5
Titanium	2.0	3.0[1)]	5.71	2.0
Platinum	4.0	2.4	5.40	4.0
Gold	4.0	2.6	5.40	4.1

1) These values were simply estimated by the linear model given in equation (69).

much smaller than the contribution of the other interactions analyzed beforehand. Clifford M. Will used the following approximation in the Weinberg-Salam model for the weak and electromagnetic interactions [50]

$$\zeta^{\mathrm{W}} = 2.2 \times 10^{-8} \frac{NZ}{A^2} \left[1 + g(N,Z)\right], \tag{74}$$

where

$$g(N,Z) = 0.295 \left[\frac{(N-Z)^2}{2NZ} + 4\sin^2\theta_W + \frac{Z}{N}\sin^2\theta_W (2\sin^2\theta_W - 1) \right]. \tag{75}$$

As can be seen from the $(N-Z)^2$ dependence, the weak interactions mediates the transition to the energetically most favorable number of neutrons and protons via beta decay. The angle $\theta_W \sim 20^\circ$ is referred to as the "Weinberg" angle. Evaluating the equations leads to

$$\zeta^{\mathrm{W}} = 2.5 \times 10^{-8} \frac{NZ}{A^2} + 3.2 \times 10^{-9} \left(\frac{N-Z}{A}\right)^2 - 2.0 \times 10^{-9} \left(\frac{Z}{A}\right)^2, \tag{76}$$

which yields values of the order of 6×10^{-9} for atoms (see equation (79)). The energy can be calculated by

$$E^{\mathrm{W}} = 23 \frac{NZ}{A} + 3.0 \frac{(N-Z)^{2\,2}}{A} - 1.9 \frac{Z^2}{A} \ \mathrm{eV}, \tag{77}$$

and yields the following values for the substances of the Hughes-Drever experiments

$$\begin{aligned} &E^{\mathrm{W}}_{^3\mathrm{He}} = 15.8 \ \mathrm{eV}, \ E^{\mathrm{W}}_{^7\mathrm{Li}} = 39.7 \ \mathrm{eV}, \ E^{\mathrm{W}}_{^{129}\mathrm{Xe}} = 730 \ \mathrm{eV}, \\ &E^{\mathrm{W}}_{^{199}\mathrm{Hg}} = 561 \ \mathrm{eV}, \ E^{\mathrm{W}}_{^{201}\mathrm{Hg}} = 566 \ \mathrm{eV}, \end{aligned} \tag{78}$$

The fractional energy contribution from the weak interaction for the substances used in the Eötvös experiments are given by

$$\begin{aligned} &\zeta^{\mathrm{W}}_{\mathrm{Be}} = 6.11 \times 10^{-9}, \ \zeta^{\mathrm{W}}_{\mathrm{Al}} = 6.12 \times 10^{-9}, \ \zeta^{\mathrm{W}}_{\mathrm{Ti}} = 6.12 \times 10^{-9}, \\ &\zeta^{\mathrm{W}}_{\mathrm{Pt}} = 6.04 \times 10^{-9}, \ \zeta^{\mathrm{W}}_{\mathrm{Au}} = 6.05 \times 10^{-9}, \end{aligned} \tag{79}$$

and it is obvious that this contribution is the smallest of all the contributions considered in this section.

Table 7: Limits on η^A for the different energy contributions arising from the strong interaction. We can not infer limits on the violation of the volume energy, since the expression which we used for the volume energy is constant for all the test bodies (see equation (66)).

Experiment	$\lvert\eta^{\mathbf{Sf}}\rvert \leq$	$\lvert\eta^{\mathbf{As}}\rvert \leq$	$\lvert\eta^{\mathbf{P}}\rvert \leq$	$\lvert\eta^{\mathbf{S}}\rvert \leq$
Princeton	7.8×10^{-9}	2.5×10^{-8}	–	1.1×10^{-8}
Moscow	4.1×10^{-10}	1.3×10^{-9}	2.0×10^{-9}	4.6×10^{-10}
Eöt-Wash (Be/Ti)	5.6×10^{-11}	1.6×10^{-9}	1.3×10^{-10}	3.8×10^{-11}
Eöt-Wash (Be/Al)	7.4×10^{-11}	7.4×10^{-10}	–	6.8×10^{-11}
MICROSCOPE	6.9×10^{-12}	1.6×10^{-11}	1.3×10^{-11}	3×10^{-10}
MICROSCOPE[1)]	5.3×10^{-13}	1.2×10^{-12}	9.9×10^{-13}	2×10^{-11}

1) If they reach their aim of $|\eta(\mathrm{Ti}, \mathrm{Pt})| \leq 10^{-15}$.

6 Limits on the Strength of the Violation of the Principle of Equivalence

We are now able to calculate limits on the WEP and LLI violation parameters η^A and δ^A from equations (13) and (17), respectively, for a certain form of energy A. We will need expressions for the energies E^A and fractional energy contributions ζ^A as well as an upper limit on the value of the Eötös ratio η or inertial mass tensor δm_I^{ij}. Updated constraints on these parameters for a variety of forms of energy are calculated and summarized, which were not performed in such a detail to date.

6.1 Limits on the Strong Interaction

For the upper limit on the strength of the violation of the WEP by the strong interaction we get the following limit from equation (13)

$$|\eta^{\mathrm{S}}| \leq \frac{|\eta(X, Y)|}{|\zeta_X^{\mathrm{S}} - \zeta_Y^{\mathrm{S}}|}. \tag{80}$$

The results of the MICROSCOPE mission which give the following limit on the Eötvös ratio $\eta(\mathrm{Ti}, \mathrm{Pt}) \leq 1.3 \times 10^{-14}$ set the limit to $|\eta^S| \leq 3.0 \times 10^{-10}$. The lowest limit on $|\eta^{\mathrm{S}}|$ to date can be deduced from the Eöt-Wash experiment. They measured $\eta(\mathrm{Be}, \mathrm{Ti}) \leq 2.1 \times 10^{-13}$ which leads to

$$|\eta^{\mathrm{S}}| \leq \frac{|\eta(\mathrm{Be}, \mathrm{Ti})|}{|\zeta_{\mathrm{Be}}^{\mathrm{S}} - \zeta_{\mathrm{Ti}}^{\mathrm{S}}|} \leq 3.8 \times 10^{-11}. \tag{81}$$

If the MICROSCOPE mission reaches its aim of $\eta(\mathrm{Ti}, \mathrm{Pt}) \leq 10^{-15}$ the upper limit on $|\eta^{\mathrm{S}}|$ will be improved to $|\eta^S| \leq 2 \times 10^{-11}$. One can also make the assumption that the different energy contributions arising from the strong interaction couple differently to gravitation and therefore violate the EEP with different strengths, corresponding to different η^A's [12]. The different limits on $|\eta^{\mathrm{S}}|$ and its descendants, obtained by the Eötvös experiments which we analyzed, are summarized in Table 7 and visualized in Figure 8.

From equation (17) we get the following expression for the upper limit on the strength of the

Table 8: Limits on on a possible anisotropy of the inertial mass for the different contributions arising from the strong interaction. A limit on the parity energy term can not be set, since the substances which were used do not have a contribution from it.

Experiment	$\lvert\delta^{\mathbf{V}}_{\mathbf{aniso}}\rvert \leq$	$\lvert\delta^{\mathbf{Sf}}_{\mathbf{aniso}}\rvert \leq$	$\lvert\delta^{\mathbf{As}}_{\mathbf{aniso}}\rvert \leq$	$\lvert\delta^{\mathbf{S}}_{\mathbf{aniso}}\rvert \leq$
Hughes-Drever	1.5×10^{-24}	2.7×10^{-24}	5.2×10^{-23}	3.7×10^{-24}
Lamoreaux et al.	3.3×10^{-31}	1.8×10^{-30}	5.7×10^{-30}	4.3×10^{-31}
Allmendinger et al.	3.2×10^{-34}	1.4×10^{-33}	7.8×10^{-33}	4.3×10^{-34}

violation of LLI by the strong interaction

$$|\delta^S| \leq \frac{|\delta m_I^{ij} c^2|}{|E^S|}. \tag{82}$$

The most stringent limit on the anisotropic part can be obtained by the experiment by F. Allmendinger and collaborators, which measured $|\delta m^{ij}_{I,\mathrm{aniso}} c^2| \leq 6.7 \times 10^{-25}$ eV by comparing ^{3}He and ^{129}Xe, giving

$$|\delta^{\mathrm{S}}_{\mathrm{aniso}}| \leq 4.3 \times 10^{-34}, \tag{83}$$

This is already very strong, whereas the limit on the isotropic part is about 26 orders of magnitude less stringent. The Kennedy-Thorndike experiments only yield limits for the strength of violation of LLI by the electromagnetic interaction. In Table 8, all the limits on the strength of violation of LLI obtained by the Hughes-Drever experiments we considered are tabulated.

6.2 Limits on the Electromagnetic Interaction

The limit on the strength of violation of the WEP by the electromagnetic interaction which follows from equation (13) is given by

$$|\eta^{\mathrm{EM}}| \leq \frac{|\eta(X,Y)|}{|\zeta_X^{\mathrm{EM}} - \zeta_Y^{\mathrm{EM}}|}. \tag{84}$$

The MICROSCOPE mission yields an upper bound $|\eta^{\mathrm{EM}}| \leq 6.6 \times 10^{-12}$, which is the lowest bound on $|\eta^{\mathrm{EM}}|$ to date. As with the strong interaction, one can assume that the different contributions to the nuclear electromagnetic energy, meaning the electrostatic, magnetostatic and hyperfine energy, couple differently to gravity, and therefore belong to different WEP-violation parameters. These calculated values are summarized in Table 9 for the Eötvös experiments we considered. Limits on $|\eta^{\mathrm{EM}}|$ and $|\eta^{\mathrm{HF}}|$ are illustrated in Figure 8.

For the upper limits on the strength of violation of LLI of the electromagnetic interaction by a possible anisotropy of the inertial mass is given by

$$|\delta^{\mathrm{EM}}_{\mathrm{aniso}}| \leq \frac{|\delta m^{ij}_{I,\mathrm{aniso}} c^2|}{|E^{\mathrm{EM}}|}, \tag{85}$$

which can be deduced from equation (17). Therefore the best limit arises from the experiment performed by F. Allmendinger and his collaborators and yields the limit $|\delta^{\mathrm{EM}}_{\mathrm{aniso}}| \leq 1.66 \times 10^{-33}$. The calculated limits for the various contributions to the electromagnetic interactions are summarized in Table 10.

Table 9: Upper limits on $|\eta^A|$ for the different contributions arising from the electromagnetic interaction, set by the experiments we consider. The limits for the hyperfine and the electromagnetic interaction are illustrated in Figure 8.

Experiment	$\lvert\eta^{\mathbf{ES}}\rvert \leq$	$\lvert\eta^{\mathbf{MS}}\rvert \leq$	$\lvert\eta^{\mathbf{HF}}\rvert \leq$	$\lvert\eta^{\mathbf{EM}}\rvert \leq$
Princeton	8.9×10^{-9}	5.9×10^{-5}	4.9×10^{-6}	8.9×10^{-9}
Moscow	4.7×10^{-10}	7.1×10^{-6}	2.5×10^{-7}	4.8×10^{-10}
Eöt-Wash (Be/Ti)	1.4×10^{-10}	8.4×10^{-7}	2.2×10^{-7}	1.4×10^{-10}
Eöt-Wash (Be/Al)	2.1×10^{-10}	1.1×10^{-6}	7.8×10^{-8}	2.1×10^{-10}
MICROSCOPE	6.5×10^{-12}	10^{-7}	4.1×10^{-9}	6.6×10^{-12}
MICROSCOPE[1)]	5.0×10^{-13}	10^{-8}	3.2×10^{-10}	5.1×10^{-13}

1) If the reach their aim of $|\eta(\mathrm{Ti}, \mathrm{Pt})| \leq 10^{-15}$.

Table 10: Limits on $|\delta^A_{\mathrm{iso}}|$ for the different contributions arising from the electromagnetic interaction.

Experiment	$\lvert\delta^{\mathbf{ES}}_{\mathbf{iso}}\rvert \leq$	$\lvert\delta^{\mathbf{MS}}_{\mathbf{iso}}\rvert \leq$	$\lvert\delta^{\mathbf{HF}}_{\mathbf{iso}}\rvert \leq$	$\lvert\delta^{\mathbf{EM}}_{\mathbf{iso}}\rvert \leq$
Hughes-Drever	7.7×10^{-23}	4.7×10^{-20}	4.72×10^{-22}	6.7×10^{-23}
Lamoreaux et al.	1.4×10^{-30}	6.2×10^{-26}	2.09×10^{-28}	1.4×10^{-30}
Allmendinger et al.	1.7×10^{-33}	2.4×10^{-29}	9.84×10^{-32}	1.7×10^{-33}

Table 11: Upper bounds on $|\eta^{\mathrm{W}}|$ for the Eötvös experiments of interest. Experimental settings and differences in weak energies are included, too.

Experiment	X, Y	$\lvert\eta(X,Y)\rvert \leq$	$\lvert\zeta_X^{\mathbf{W}} - \zeta_Y^{\mathbf{W}}\rvert$	$\lvert\eta^{\mathbf{W}}\rvert \leq$
Princeton	Al, Au	2.3×10^{-11}	7×10^{-11}	3×10^{-1}
Moscow	Al, Pt	1.2×10^{-12}	8×10^{-11}	2×10^{-2}
Eöt-Wash (Be/Ti)	Be, Ti	2.1×10^{-13}	10^{-11}	2×10^{-1}
Eöt-Wash (Be/Al)	Be, Al	2.0×10^{-13}	10^{-11}	2×10^{-2}
MICROSCOPE	Ti, Pt	1.3×10^{-14}	7×10^{-11}	2×10^{-4}
MICROSCOPE[1)]	Ti, Pt	10^{-15}	7×10^{-11}	10^{-5}

1) If the reach their aim of $|\eta(\mathrm{Ti}, \mathrm{Pt})| \leq 10^{-15}$.

The scalar part of the anomalous inertial mass tensor arising from the electromagnetic interaction is limited by Kennedy-Thorndike experiments e.g. M. E. Tobar and his collaborators set $|\delta_{\mathrm{iso}}^{\mathrm{EM}}| \leq 5.7 \times 10^{-8}$ (see equation (21) and corresponding text), which is about 10^{26} times less stringent limit than the one on the anisotropic part.

6.3 Limits on the Weak Interaction

The contribution of the weak interaction to the total energy is very small (ζ_X^{W} between 10^{-9} and 10^{-8}, see equation (79)) and hence the difference between different substances is even smaller. Values for the differences and the corresponding limits on the strength of the WEP violation of the weak interaction are given in Table 11, where the best limit to date, obtained by the MICROSCOPE mission, is given by $|\eta^{\mathrm{W}}| \leq 2 \times 10^{-4}$, which can be calculated through

$$|\eta^{\mathrm{W}}| \leq \frac{|\eta(X,Y)|}{|\zeta_X^{\mathrm{W}} - \zeta_Y^{\mathrm{W}}|}. \tag{86}$$

These upper bounds are visualized in Figure 8 for the Eötvös experiments of interest.

The best limit on a possible anisotropy of the inertial mass up to now can be calculated by the expression following from equation (17)

$$|\delta_{\mathrm{aniso}}^{\mathrm{W}}| \leq \frac{|\delta m_{I,\mathrm{aniso}}^{ij} c^2|}{|E^{\mathrm{W}}|} \leq 9 \times 10^{-28}. \tag{87}$$

6.4 Summary of the Limits

The limits placed on the strength of the WEP-violation parameters for the three fundamental interactions as well as the hyperfine interaction are illustrated in Figure 8, and the most stringent limits on all possible violations of the EEP obtained by the experiments which were considered are summarized in Table 12.

Limits on the strength of the bare LLI violation for the three fundamental interactions can be computed by using equation (16) and are given by

$$|\delta_0^{\mathrm{S}}| \leq 2.8 \times 10^{-28}, \;\; |\delta_0^{\mathrm{EM}}| \leq 1.1 \times 10^{-27}, \;\; |\delta_0^{\mathrm{W}}| \leq 5.9 \times 10^{-22}. \tag{88}$$

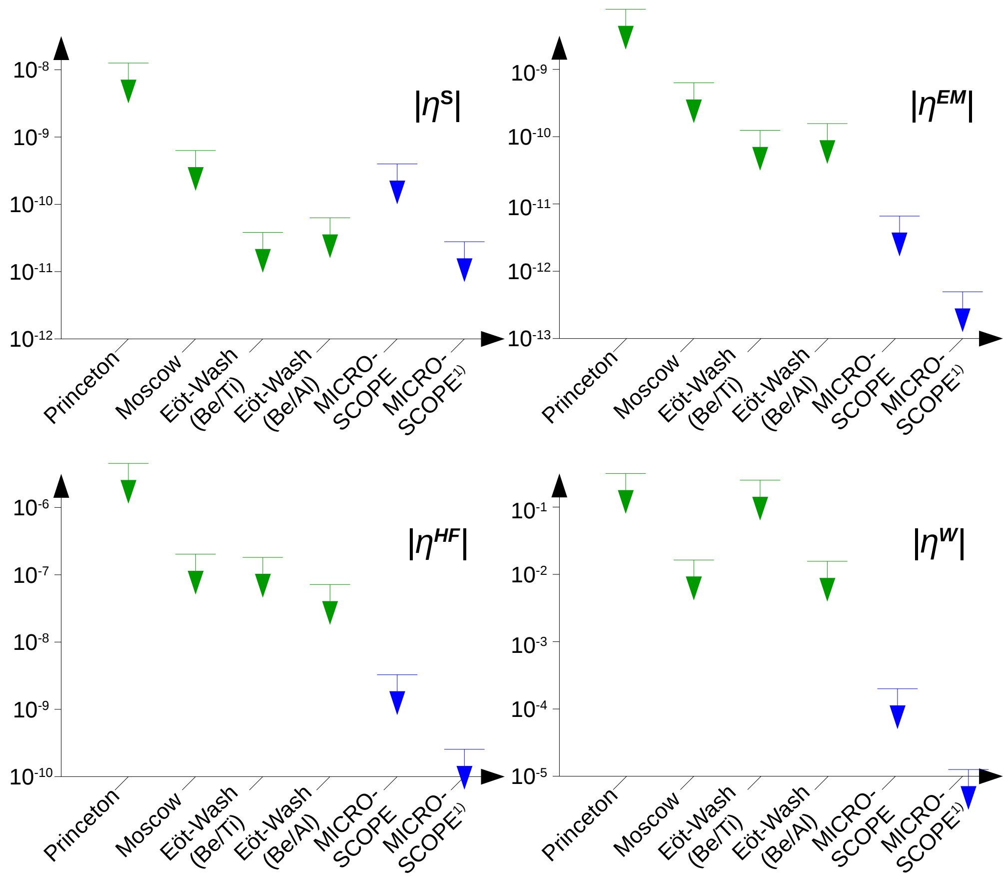

Figure 8: Limits on the WEP violation parameter for the three fundamental and the hyperfine interaction. Torsion balance experiments are given in green and experiments performed in the orbit in blue. In the case of the strong interaction, the best limit to date is obtained by the Eöt-Wash experiment, in all the other cases by the MICROSCOPE mission. 1) If the MICROSCOPE mission reaches its aim.

Table 12: Most stringent limits on the strength of violation of the EEP. Kennedy-Thorndike and gravitational redshift experiments were only performed to test the electromagnetic interaction and therefore do not yield any values for the strong or weak interaction. The limits set on the LPI violation parameter depend on the assumption one makes about which forms of energy do couple slightly non metrically to gravity and are therefore only model independently set for the electromagnetic and the hyperfine interaction.

Interaction A	$\|\eta^A\| <$	$\|\delta^A_{\mathbf{aniso}}\| <$	$\|\delta^A_{\mathbf{iso}}\| <$	$\|\alpha^A\| <$
Strong	3.8×10^{-11}	4.3×10^{-34}	–	–
Volume	-	3.2×10^{-34}	–	–
Surface	6.9×10^{-12}	1.4×10^{-33}	–	–
Asymmetry	1.6×10^{-11}	7.8×10^{-33}	–	–
Pairing	1.3×10^{-11}	–	–	–
Electromagnetic	6.6×10^{-12}	1.7×10^{-33}	5.7×10^{-8}	2×10^{-4}
Electrostatic	6.6×10^{-12}	1.7×10^{-33}	5.7×10^{-8}	–
Magnetostatic	1.3×10^{-7}	–	5.7×10^{-8}	–
Hyperfine	4.1×10^{-9}	9.8×10^{-32}	5.7×10^{-8}	2×10^{-4}
Weak	2×10^{-4}	9×10^{-28}	–	–

One can see that the limits on the weak interaction are a few orders of magnitude less stringent than limits on the other fundamental interactions and their descendants (except the gravitational interaction), which is due to their small contribution to the total energy of the test bodies.

In the next section we will calculate limits on the EEP violation-parameters under the assumption that Schiff's conjecture is valid and therefore equations (39) and (51), or more generally equation (54), holds true.

7 New Limits Obtained by Schiff's Conjecture

In this section we will assume the validity of Schiff's conjecture and use equation (54) which describes the quantitative relationship between violations of the three assumptions of the EEP

$$\eta(X,Y) = \alpha^A(\zeta^A_X - \zeta^A_Y) - \delta^B_{\text{iso}}(\zeta^B_X - \zeta^B_Y). \tag{89}$$

As can be seen from this expression, there exists the option that the two terms exactly compensate each other. For example if both violations are due to a non metric coupling of the same form of energy $A = B$ and the same strength $\alpha^A = \delta^B_{\text{iso}}$. This would lead to the fact that the statement: "WEP directly implies the EEP", is wrong. However, this would simply be a coincidence, as there does not exist any physical reason for this option. Due to this fact we will not consider a violation of both LLI and LPI at the same time, but assume that LLI is valid at first, and that LPI is valid afterwards[17]. Updated constraints on the different EEP-violation parameters will be calculated under the assumption that Schiff's conjecture is correct, as was not done in such detail before.

[17] In fact, we do not have to make the stringent assumption that LLI or LPI is strictly valid. It is enough to assume that the strength of its violation is a few orders of magnitude weaker such that it can be neglected

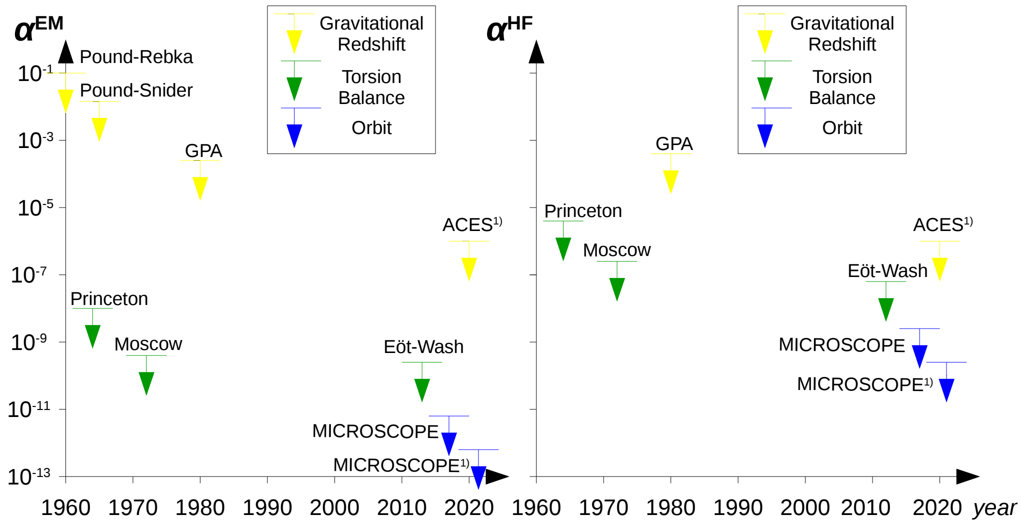

Figure 9: Limits on the strength of violation of LPI under the assumption that all the electromagnetic violates LPI (right) and that all the hyperfine energy violates LPI (left). It can be clearly seen that the limits obtained by the Eötvös experiments are ways more stringent than the ones from the standard gravitational redshift experiments in both of the models.

7.1 New Limits on the Strength of Violation of Local Position Invariance Set by Eötvös experiments

Limits on α arising from Eötvös experiments depend on the form of energy which is transferred and on the assumption we make which forms of energy A violate the WEP. From equation (39) we get the following expression

$$|\alpha^A| = \frac{|\eta(X,Y)|}{|\zeta_X^A - \zeta_Y^A|}, \tag{90}$$

by assuming that LLI is valid or the strength of its violation is at least a few orders of magnitude smaller.

7.1.1 Standard Gravitational Redshift Experiment

In a molecular clock, the energy transferred is mainly nuclear electrostatic energy. So if we assume a model in which all the electrostatic nuclear energy violates the WEP by a slight non metric coupling to gravity, the upper bounds on the LPI violation parameter for the electrostatic energy $|\alpha^{\mathrm{ES}}|$ can be calculated by using the expression given above

$$|\alpha^{\mathrm{ES}}| = \frac{|\eta(X,Y)|}{|\zeta_X^{\mathrm{ES}} - \zeta_Y^{\mathrm{ES}}|}. \tag{91}$$

This calculation was already done by Kenneth L. Nordtvedt in 1975, at a time at which the most stringent limit on the Eötvös ratio was set by the Moscow experiment with $|\eta(\mathrm{Al},\mathrm{Pt})| \leq$

1.2×10^{-12}. If we redo this calculation we therefore arrive at $|\alpha^{\mathrm{ES}}| \leq 4.7 \times 10^{-10}$. The value obtained by Kenneth L. Nordtvedt yielded a limit of $|\alpha^{\mathrm{ES}}| \leq 2.5\times10^{-10}$ [33] which lies in the same range of magnitude. The deviation occurred due to different models for the fractional energy contribution of the nuclear electrostatic energy. Also he used the difference between aluminum and gold $\zeta^{\mathrm{ES}}_{\mathrm{Al}} - \zeta^{\mathrm{ES}}_{\mathrm{Au}}$, whereas he should have used the difference between aluminum and platinum $\zeta^{\mathrm{ES}}_{\mathrm{Al}} - \zeta^{\mathrm{ES}}_{\mathrm{Au}}$[18]. To date, using the more stringent limits obtained by the MICROSCOPE mission, the upper bound on the LPI violation parameter of the electrostatic energy can be reduced to $|\alpha^{\mathrm{ES}}| \leq 6.5 \times 10^{-12}$, or if they reach their aim, even lower to $|\alpha^{\mathrm{ES}}| \leq 5.0 \times 10^{-13}$. All of these limits are ways more stringent than what one expects to reach by performing a standard gravitational red-shift experiment with a molecular clock.

In a model in which is assumed that only energy arising from the hyperfine interaction has a non metric coupling to gravity, the relation between its LPI violation parameter and the Eötvös ratio is given by

$$|\alpha^{\mathrm{HF}}| = \frac{|\eta(X,Y)|}{|\zeta^{\mathrm{HF}}_X - \zeta^{\mathrm{HF}}_Y|}. \tag{92}$$

This is a particularly interesting model, since all the clocks used in the most recent standard gravitational redshift experiments (e.g. hydrogen masers ot cesium atomic clocks) depend on transition in hyperfine energy levels and therefore set limits on $|\alpha^{\mathrm{HF}}|$, too. This calculation was performed by Kenneth L. Nordtvedt [33] with the results from the Moscow experiment, too, and yielded a limit $|\alpha^{\mathrm{HF}}| \leq 2 \times 10^{-8}$. Again he used the difference in fractional energy contribution between aluminum and gold, whereas he should have used aluminum and platinum (see footnote 18). Redoing his calculation gives an upper bound of $|\alpha^{\mathrm{HF}}| \leq 2.5 \times 10^{-7}$, which is about an order of magnitude less stringent. This difference arises from the fact that he approximated the value of $|\delta M/M| = 5 \times 10^{-5}$ for the fractional mass difference, whereas we calculated $\zeta^{\mathrm{HF}}_{\mathrm{Al}} - \zeta^{\mathrm{HF}}_{\mathrm{Pt}} \leq 4.7 \times 10^{-6}$. The most stringent limit in this model to date is set by the MICROSCOPE mission, which fixes $|\alpha^{\mathrm{HF}}| \leq 4.1 \times 10^{-9}$. This value is 50'000 times more stringent than the one obtained by the GPA experiment and and already about 1500 times more accurate than the limits one hopes to reach using the ACES mission. If the MICROSCOPE mission reaches its aim and gets to $|\eta(\mathrm{Ti},\mathrm{Pt})| \leq 10^{-15}$ the limit on the LPI violation parameter is even lowered to $|\alpha^{\mathrm{HF}}| \leq 3.2 \times 10^{-10}$ which is about 600'000 times more stringent than the lowest bound on $|\alpha^{\mathrm{HF}}|$ obtained by standard gravitational redshift experiments to date. The different upper bounds set from standard gravitational redshift as well as Eötvös experiments can be seen in Figure 9.

In a model in which we assume that all the electromagnetic energy violates the WEP by non metric coupling to gravity, we get the relation between the LPI violation parameter and the Eötvös ratio from equation (90)

$$|\alpha^{\mathrm{EM}}| \leq \frac{|\eta(X,Y)|}{|\zeta^{\mathrm{EM}}_X - \zeta^{\mathrm{EM}}_Y|}, \tag{93}$$

which can be used to set limits on all experiments which use a clock transferring electromagnetic energy. The best limits set by standard gravitational redshift experiments are therefore the ones by the GPA experiment and will by improved by the ACES mission. From the Eötvös experiments, namely the MICROSCOPE mission, we get $|\alpha^{\mathrm{EM}}| \leq 6.5 \times 10^{-12}$ which is more than seven orders of magnitude more stringent than the value obtained by the GPA mission and therefore yields the most stringent result on $|\alpha^{\mathrm{EM}}|$ to date. It is about six orders of magnitude more stringent than the results one hopes to find by the ACES mission, too. The limits on the LPI violation parameter obtained by the different experiments under the assumption that either

[18] Aluminum and gold were used as substances in the Princeton experiment, but the Moscow experiment, which could set a more stringent limit on the Eötvös ratio, used aluminum and platinum.

all the electromagnetic energy or all the hyperfine energy violates LPI and thus the WEP are illustrated in Figure 9.

The relation given in equation (90) can generally be used for all forms of energy including the strong and the weak interaction. This means we can obtain limits on the strength of violation of LPI by the strong and weak interaction from the Eötvös experiments, where there were no limits set from standard gravitational redshift experiments up to date. Most stringent limits obtained in this way are summarized in Table 13.

7.1.2 "Null" Experiment

For the result of a "null" experiment (comparison between two different clocks at the same position) the relation which can be deduced from equation (90) is given by

$$|\alpha^A - \alpha^B| = \left| \left(\frac{1}{\zeta_X^A - \zeta_X^B} - \frac{1}{\zeta_Y^A - \zeta_Y^B} \right) \eta(X,Y) \right|, \tag{94}$$

where A and B are the types of energy transferred in the two clocks, respectively. This means we can only find a quantitative relationship if the two energies transfer different forms of energy. For example the comparison of a hydrogen maser, which depends on the transition between hyperfine levels, and a SCSO clock, which transfers electromagnetic energy, yields the following limit $|\alpha_{\mathrm{H}}^{\mathrm{HF}} - \alpha_{\mathrm{SCSO}}^{\mathrm{EM}}| \leq 1.7 \times 10^{-2}$. We can infer the following limit from the MICROSCOPE mission by simply setting $A = \mathrm{HF}$ and $B = \mathrm{EM}$ in equation (94) $|\alpha_{\mathrm{H}}^{\mathrm{HF}} - \alpha_{\mathrm{SCSO}}^{\mathrm{EM}}| \leq 3.3 \times 10^{-12}$. This limit is nearly ten orders of magnitude more stringent than the value obtained by the null experiment. The aim of the MICROSCOPE mission would set the upper bound even lower to a value of $|\alpha_{\mathrm{H}}^{\mathrm{HF}} - \alpha_{\mathrm{SCSO}}^{\mathrm{EM}}| \leq 2.5 \times 10^{-13}$. We can therefore state that "null" experiments which use clocks transferring different forms of energy are already confirmed by Eötvös experiments up to a very accurate level.

7.2 Possible New Limits on the Violation of the Weak Equivalence Principle from Gravitational Redshift Experiments

As seen in the previous subsection, in a model in which the only energy that violates the WEP is the hyperfine energy, one can infer a limit on α^{HF} from the Eötvös ratio. This can be done the other way around, too, namely to place limits on η^{HF} from the gravitational redshift experiment by using equation (41) which yields the following relation

$$|\eta^{\mathrm{HF}}| \leq |\alpha^{\mathrm{HF}}|. \tag{95}$$

We get an upper bound $|\eta^{\mathrm{HF}}| \leq 2 \times 10^{-4}$ from the GPA experiment. This limit is more than four orders of magnitude less stringent than the best ones obtained by Eötvös experiments. Generally speaking, the Eötvös experiment seem to be more accurate on the determination of the WEP and LPI violation parameters η^{A} and α^{A}. However, if one would use a clock which depends on a transition of the weak interaction, for example a clock which depends on the beta-decay rate, we could use the following relationship from equation (39)

$$|\alpha^{\mathrm{W}}(\zeta_X^{\mathrm{W}} - \zeta_Y^{\mathrm{W}})| \simeq |\eta(X,Y)|, \tag{96}$$

to set limits on the Eötvös ratio. To reach e.g a limit of $|\eta| \leq 10^{-15}$ would require an upper bound $|\alpha^{\mathrm{W}}| \leq 1.4 \times 10^{-5}$, which could be reached without too much effort in the near future. Clearly this relationship only holds true in a model in which the weak interaction is the only from of energy which couples non metrically to gravity. One could also directly use the expression following from equation (41)

$$|\eta^{\mathrm{W}}| \leq |\alpha^{\mathrm{W}}|, \tag{97}$$

and as the currently lowest limit lies at $|\eta^{\mathrm{W}}| \leq 2 \times 10^{-4}$, a gravitational redshift experiment could improve it. Therefore a gravitational redshift experiment using a clock which transfers weak energy would be a very preferable thing to do in the near future, as was already proposed by Kenneth L. Nordtvedt in 1975 at the end of his analysis [33]. However, one must be cautious since all the proposals depend on the assumption that only the weak interaction couples non metrically to gravity and therefore violates the EEP. Also there could be a contribution to the violation of the WEP from violations of LLI, which can not be excluded by gravitational redshift experiments, too.

7.3 New Limits on the Violation of Local Lorentz Invariance from Eötvös Experiments

If we assume that LPI is fulfilled, or that the strength of its violation is at least a few orders of magnitude less, we can use the following relation found in equation (51)

$$|\delta^{A}_{\mathrm{iso}}| = \frac{|\eta(X,Y)|}{|\zeta^{A}_{X} - \zeta^{A}_{Y}|}. \tag{98}$$

The most stringent restrictions on the isotropic or scalar part of the anomalous inertial mass tensor arises not from direct tests of the LLI, but from the Eötvös experiments. This was already stated by Mark P. Haugan and Clifford M. Will in 1987, who calculated the upper bound on a scalar deviation of the inertial mass from Eötvös experiments[19] [21]. Since the only experiments considered in this thesis, which directly set limits on the isotropic part are the Kennedy-Thorndike experiments, which only set limits on the electromagnetic interaction. We will have a look at the relation between a scalar deviation of the inertial mass and the Eötvös experiment of the electromagnetic interaction

$$|\delta^{\mathrm{EM}}_{\mathrm{iso}}| = \frac{|\eta(X,Y)|}{|\zeta^{\mathrm{EM}}_{X} - \zeta^{\mathrm{EM}}_{Y}|}. \tag{99}$$

The MICROSCOPE mission yields the most stringent limit on the violation of the isotropic part of LLI to date $|\delta^{\mathrm{EM}}_{\mathrm{iso}}| \leq 6.6 \times 10^{-12}$. This is about four orders of magnitude more accurate than the best limit set by Kennedy-Thorndike experiments (see e.g. M. E. Tobar et al.). Upper limits on $\delta^{\mathrm{EM}}_{\mathrm{iso}}$ and $\delta^{\mathrm{HF}}_{\mathrm{iso}}$ obtained from experiments mentioned are represented in Figure 10. Moreover we can now use this relation to infer limits on the scalar parts of the anomalous inertial mass tensor for all forms of energy, not only the electromagnetic Table 13 summarizes all the new limits which can be obtained by the validity of Schiff's conjecture (in comparison to Table 12). From the analysis in this section, one might say that if Schiff's conjecture is valid, except for a possible anisotropy of the inertial mass, all aspects of the EEP are tested in the most stringent way by Eötvös experiments. It therefore makes sense to put more effort in new and more accurate tests of the WEP, which seems to confirm the statement that the WEP directly implied the EEP.

[19] The most stringent limit arising from Eötvös experiments at this time were set by the Moscow experiment, which measured $|\eta| \leq 10^{-12}$. Using this value, they calculated a limit of $|\delta| \leq 10^{-9}$ on a possible scalar mass anomaly.

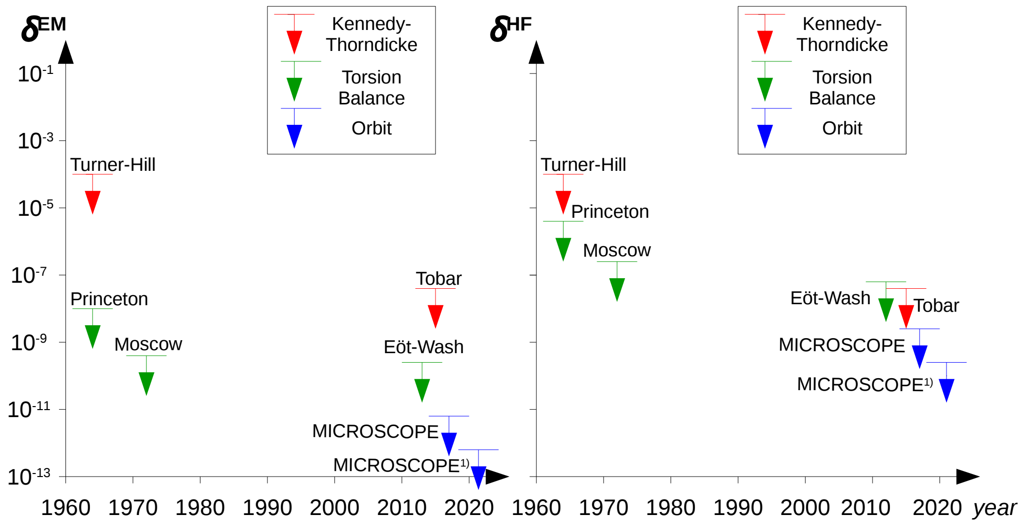

Figure 10: Limits on the strength of violation of LLI under the assumption that all the electromagnetic (left) and all the hyperfine energy violates LLI, respectively. In the first case, the Eötvös experiments set much more stringent limits than the Kennedy-Thorndike experiments, about four to six orders of magnitude. Whereas in the second case they are more stringent, but the Kennedy-Thorndike experiments can still compete with the Eötvös experiments.

Table 13: Most stringent limits on the strength of violation of different aspects of the EEP obtained by assuming that Schiff's conjecture is valid. All the most stringent limits on η^A, $\delta^A_{\rm iso}$ and α^A are obtained by Eötvös experiments. Only the most stringent limits on $\delta^A_{\rm aniso}$ are set by Hughes-Drever experiments.

Interaction A	$\lvert\eta^A\rvert \leq$	$\lvert\delta^A_{\bf aniso}\rvert \leq$	$\lvert\delta^A_{\bf iso}\rvert \leq$	$\lvert\alpha^A\rvert \leq$
Strong	3.8×10^{-11}	4.3×10^{-34}	3.8×10^{-11}	3.8×10^{-11}
Volume	-	3.2×10^{-34}	–	–
Surface	6.9×10^{-12}	1.4×10^{-33}	6.9×10^{-12}	6.9×10^{-12}
Asymmetry	1.6×10^{-11}	7.8×10^{-33}	1.6×10^{-11}	1.6×10^{-11}
Pairing	1.3×10^{-11}	–	1.3×10^{-11}	1.3×10^{-11}
Electromagnetic	6.6×10^{-12}	1.7×10^{-33}	6.6×10^{-12}	6.6×10^{-12}
Electrostatic	6.6×10^{-12}	1.7×10^{-33}	6.6×10^{-12}	6.6×10^{-12}
Magnetostatic	1.3×10^{-7}	–	1.3×10^{-7}	1.3×10^{-7}
Hyperfine	4.1×10^{-9}	9.8×10^{-32}	4.1×10^{-9}	4.1×10^{-9}
Weak	2×10^{-4}	9×10^{-28}	2×10^{-4}	2×10^{-4}

8 The $TH\epsilon\mu$ Formalism

This section will give a few aspects of the $TH\epsilon\mu$ formalism, following section 2.2.2 in Clifford M. Wills living review "The Confrontation betweeen General Relativity and Experiment" [51], which applies the formalism to a relation between Eötvös and gravitational redshift experiments. We will in the following redo the calculations of the previous section using this formalism and compare the results to the ones we obtained beforehand.

Alan P. Lightman and David L. Lee developed a framework called the $TH\epsilon\mu$ formalism, which they used to give a restricted proof of Schiff's conjecture [26]. It is in fact restricted to charged particles in an external static spherically symmetric gravitational field, and can therefore only describe the electromagnetic interaction. The motion of particles is characterized by two functions $T(U)$ and $H(U)$ and the response of the electromagnetic field by $\epsilon(U)$ and $\mu(U)$. Every metric theory satisfies

$$\epsilon = \mu = \left(\frac{H}{T}\right)^{1/2}, \tag{100}$$

for all U. The non-metric parameters Γ_0 and Λ_0, which give a measure for violations of LPI, are defined by the following

$$\Gamma_0 = -c_0^2 \frac{\partial}{\partial U} \ln[\epsilon(T/H)^{1/2}]_0, \;\; \Lambda_0 = -c_0^2 \frac{\partial}{\partial U} \ln[\mu(T/H)^{1/2}]_0, \tag{101}$$

respectively. The parameter which signals violations of LLI is defined by

$$\Upsilon_0 = 1 - \left[\frac{T}{H}\epsilon\mu\right]_0 = -\delta. \tag{102}$$

The validity of the EEP implies that $\Gamma_0 = \Lambda_0 = \Upsilon_0 = 0$ everywhere. The acceleration of a freely falling spherical composite test body X of electromagnetically interacting particles can be written as [51]

$$a_X = \frac{m_G}{m_X} g = \left[1 + \zeta_X^{\mathrm{ES}}\left(2\Gamma_0 - \frac{8}{3}\Upsilon_0\right) + \zeta_X^{\mathrm{MS}}\left(2\Lambda_0 - \frac{4}{3}\Upsilon_0\right) + ...\right] g. \tag{103}$$

If we set $\Upsilon_0 = 0$ because of the very tight constraints we have from test of LLI, we arrive at the following relations

$$|\eta^{\mathrm{ES}}| = |2\Gamma_0|, \;\; |\eta^{\mathrm{MS}}| = |2\Lambda_0|. \tag{104}$$

From the most stringent limits on the Eötvös ratio η, obtained by the MICROSCOPE mission, we can infer the limits on the non-metric parameters $|\Gamma_0| \leq 3.3 \times 10^{-12}$ and $|\Lambda_0| \leq 6.5 \times 10^{-8}$. From the gravitationally modified Dirac equation, an expression for the gravitational redshift can be inferred, which gives $\alpha^{\mathrm{HF}} = -3\Gamma_0 + \Lambda_0$, for a hyperfine transition clock, such as a hydrogen maser or a cesium clock, and $\alpha^{\mathrm{EM}} = -\frac{3}{2}\Gamma_0 + \frac{1}{2}\Lambda_0$, for an electromagnetic transition clock, eg. an SCSO clock. The limits on $|\alpha^A|$ set by the Eötvös experiments are therefore given by $|\alpha^{\mathrm{HF}}| \leq 6.5 \times 10^{-8}$ and $|\alpha^{\mathrm{EM}}| \leq 3.3 \times 10^{-8}$, respectively. The limit on a clock comparison experiment between a hydrogen maser and an SCSO clock can be obtained by $|\alpha_{\mathrm{H}} - \alpha_{\mathrm{SCSO}}| = \frac{3}{2}|\Gamma_0 - \Lambda_0|$ which leads to $|\alpha_{\mathrm{H}} - \alpha_{\mathrm{SCSO}}| \leq 9.8 \times 10^{-8}$. If we compare these constraints to the ones we calculated in the previous section, we realize that the limits calculated in this framework are not as stringent as the ones we calculated, which could be due to the large impact the magnetostatic energy has in this formalism. An exact analysis of the origin of these deviations is beyond the scope of this thesis.

9 Conclusions

The first thing one might say by only having a short glimpse at Table 12, which presents the most stringent limits, is that it would in any case, irrespective of any other assumptions one may want to verify, be very useful and necessary to perform tests of the EEP in such a way that they test for violations of any form of energy. Both Kennedy-Thorndike and gravitational redshift experiments only place limits on the strength of violation of the electromagnetic interaction, whereas it would be of much more use to include experimental setups which test for violations of the EEP by the strong or the weak interaction as well.

If one assumes Schiff's conjecture to be valid, then the findings clearly indicate that besides the tests for any possible anisotropy of the inertial mass, namely the Hughes-Drever experiments, the Eötvös experiments, and especially its most accurate experiment performed to date by the MICROSCOPE mission, can be regarded as the central tests of the EEP. A closer look on the values in Table 13, which presents the most stringent limits obtained by using Schiff's conjecture, reveals that three of the four different parameters are most stringently constraint by its results. It is the most accurate test of all the three pillars of the EEP at the same time and it is therefore highly important to further improve upper bounds on the strength on a possible violation of the WEP, as planned missions such as GG [31] or STEP [34] are going to do. Another motivation to advocate such experiments is the fact that modern theories of quantum gravity, such as string theory or loop-quantum gravity predict violations of the WEP at some level. Upper limits on the strength of a possible violation of LPI by the electromagnetic or hyperfine interaction, which will be tested by the ACES mission up to parts in 10^6, are in fact already dismissed by the current limits obtained by the MICROSCOPE mission, which set them to parts in 10^{12} and 10^9, respectively. The chance to measure any deviation from the EEP is exceedingly small since it would at the same time falsify Schiff's conjecture and general assumptions on which it is formulated, such as the principle of energy conservation. The same argument holds true for any future "null" gravitational redshift experiments which use clocks that depend on a transition of a different form of energy. The upper bound set from the MICROSCOPE mission to date is already at parts in 10^{12}, whereas the null experiments are currently at parts in 10^2. For tests of a scalar mass anomaly of the inertial mass, such as e.g. Kennedy-Thorndike experiments, it is still possible to compete with constraints set by Eötvös experiments. It would be most useful again, to perform these tests with an emphasis on any possible violations of LLI scaled on the strong or the weak interaction. Especially the upper bounds on a possible violation of the EEP by the weak interaction are only of the order of parts in 10^4.

A further indication is that, as the validity of Schiff's conjecture is still questioned (and probably always will be), tests of the three pillars of the EEP should either way be performed by various experiments up to the most accurate results which are possible to obtain, even though a deviation may not be expected in many models.

Outlook: A very promising task would be to relate all the parameters of the SME (see section 3.2) to the results of the Eötvös experiments via Schiff's conjecture and find the most stringent constraints on any possible violation of the EEP this way.

It would also be of interest to relate experiments, as well as their results, performed to test the SEP, to each other. It is highly indicative that tests of the GEWP (such as lunar laser ranging) play the same central role in testing the SEP, as the standard Eötvös experiments do in testing the EEP.

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List of Tables

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List of Symbols and Abbreviations

Latin

Symbol	**Meaning**	**Definition**
A	Atomic Mass Number	Section 5
A	Form of Energy	Section 3.1
a	Acceleration	Section 3.1
a_X	Acceleration of X	Equation (7)
c	Speed of Light	Section 2.1
c_0	Universal Speed of Light	Section 3.2
E^A	Energy of Form A	Section 3.1
E^{B}	Binding Energy	Section 3.2
E_{tot}	Total Energy	Section 4.2
G	Gravitational Constant	Section 2.1
g	Gravitational Acceleration	Section 3.1
$g_{\mu\nu}$	Metric Tensor	Section 2.2
H	Random Function	Section 8
h	Height	Section 4.1
m	Mass	Section 2.1
m_0	Sum of Constituent Masses	Section 4.2
m_I	Inertial Mass	Section 2.1
δm_I^{ij}	Anomalous Inertial Mass Tensor	Equation (15)
$\delta m_{I,\mathrm{aniso}}^{ij}$	Anisotropic Parts of the Anomalous Inertial Mass Tensor	Section 3.2
$\delta m_{I,\mathrm{iso}}^{ij}$	Scalar Parts of the Anomalous Inertial Mass Tensor	Section 3.2
m_G	Gravitational Mass	Section 2.1
N	Neutron Number	Section 5
P_{KT}	Kennedy-Thorndike Parameter	Section 3.3
r	Size of the Test Body	Section 2.1
T	Random Function	Section 8
U	Gravitational Potential	Section 3.3
ΔU	Difference in Gravitational Potential	Section 3.3
u	Function	Section 8
v	Velocity	Section 3.3
v_X	Velocity of Test Body X	Section 4.1
w	Velocity of the Preferred Frame	Section 3.2
X	Test Body	Section 3.1
x	Coordinate	Section 2.2
Z	Atomic Number	Section 5
z	Gravitational Redshift	Equation (22)
z	Altitude	Section 4.1

Greek

Symbol	Meaning	Definition
α	LPI Violation Parameter	Section 3.3
α^A	LPI Violation Parameter of Energy A	Section 4.1
Γ_0	Non-Metric Parameter	Section 8
$\Gamma\kappa_{\mu\nu}$	Christoffel Symbol	Section 2.2
δ	LLI Violation Parameter	Equation (20)
δ^A	LLI Violation Parameter of Energy A	Section 3.2
δ_0^A	Bare LLI Violation Parameter	Equation (16)
δ_{iso}^A	Isotropic Part of the LLI Violation Parameter	Section 4.2
$\delta_{\mathrm{aniso}}^A$	Isotropic Part of the LLI Violation Parameter	Section 6.1
ϵ	Random Function	Section 8
η	Eötvös ratio	Equation (11)
$\eta(X,Y)$	Eötvös Ratio of X and Y	Equation (11)
η^A	WEP violation Parameter	Section 3.1
Λ_0	Non-Metric Parameter	Section 8
λ	Wavelength	Section 3.3
$\Delta\lambda$	Shift in Wavelength	Section 3.3
μ	Random Function	Section 8
ν	Frequency	Section 3.3
$\Delta\nu$	Shift in Frequency	Section 3.3
σ	Self-Gravity Parameter	Equation (2)
τ	Proper Time	Section 2.2
Υ_0	LLI Violation Parameter	Section 8
ζ	Fractional Energy Contribution	Equation (8)
ζ_X^A	Fractional Contribution of Energy of Form A of X	Equation (8)

Abbreviation

Abbreviation	Meaning	Definition
ACES	Atomic Clock Ensemble in Space	Section 3.3
AEgIS	Antimatter Experiment: Gravity, Interferometry, Spectroscopy	Section 3.1
CNES	Centre National d'Études Spatiales	Section 3.3
EEP	Einstein Equivalence Principle	Section 1
EP	Principle of Equivalence	Section 1
GG	Galileo Galilei	Section 3.1
GR	General Theory of Relativity	Section 1
GPA	Gravity Probe A	Section 3.3
GWEP	Gravitational Weak Equivalence Principle	Section 2.3
LLI	Local Lorentz Invariance	Section 1
LPI	Local Position Invariance	Section 1
PHARAO	Projet d'Horologe Atomique par Refroidissement d'Atomes en Orbit	Section 3.3
SCSO	Superconducting-Cavity Stabilized Clock	Section 3.3
SEP	Strong Equivalence Principle	Section 1
SME	Standard Model Extension	Section 3.2
STEP	Satellite Test of the Equivalence Principle	Section 3.1
UFF	Universality of the Free Fall	Section 1
WEP	Weak Equivalence Priniciple	Section 1
MICROSCOPE	MICRO-Satellite á traînée Compensée pour l'Observation du Principe d'Equivalence	Section 1